Metric Spaces

How can we define distance between two real numbers $a, b \in \mathbb{R}$?

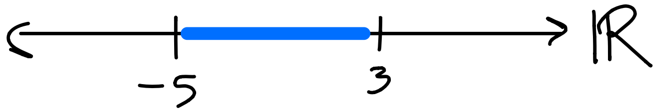
When we think of distance, we think of positive numbers.

$$\boxed{\text{Goal 1: } d(a,b) > 0}$$

Actually... Can distance be zero? Intuitively, yes. The distance between a number and itself is zero.

$$\boxed{\text{Goal 1 (modified): } d(a,b) > 0 \text{ if } a \neq b \text{ and } d(a,b) = 0 \text{ if } a = b}$$

Example: $a = -5$ and $b = 3$



$$d(-5,3) = 3-(-5)$$
$$= 3+5$$
$$= 8$$

Idea 1: For $a, b \in \mathbb{R}$, $d(a,b) = b-a$

When we think of distance, we also expect that starting point doesn't matter, ie

$$\boxed{\text{Goal 2: } d(a,b) = d(b,a)}$$

$$d(3,-5) = -5-3$$
$$= -8$$

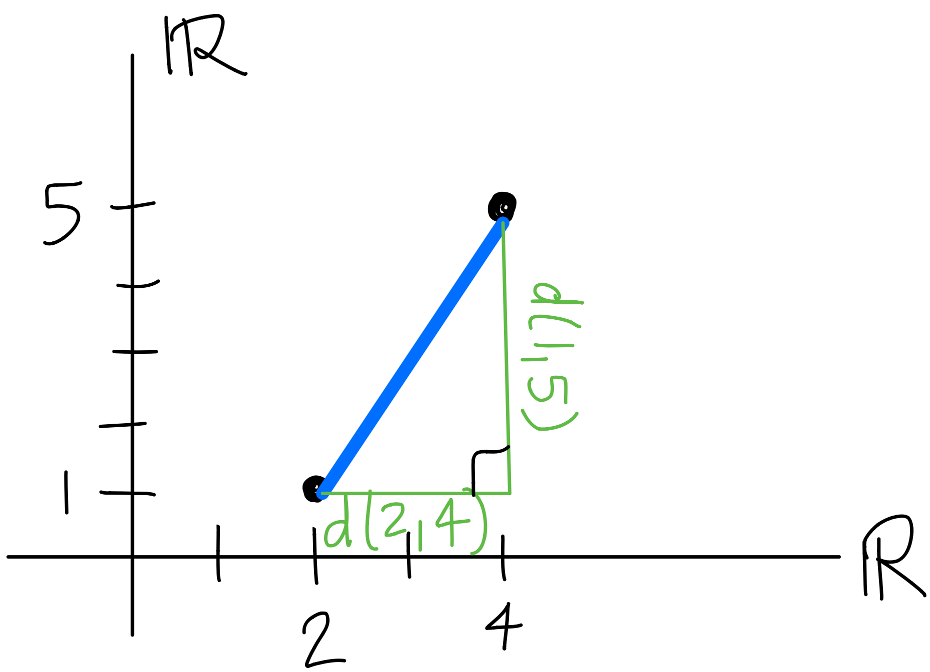
Idea 1 violates Goal 2 and Goal 1!

How can we fix this? Introduce absolute value!

Idea 2: For $a, b \in \mathbb{R}$, $d(a,b) = |a-b|$

How can we define distance between two points in the plane $\mathbb{R}^2$?

Example: $(2,1)$ and $(4,5)$



$$d((2,1),(4,5))^2 = d(2,4)^2 + d(1,5)^2$$

$$d((2,1),(4,5)) = \pm\sqrt{d(2,4)^2 + d(1,5)^2}$$

(Goal 1: choose plus)

$$d((2,1),(4,5)) = \sqrt{d(2,4)^2 + d(1,5)^2}$$

$$= \sqrt{|2-4|^2 + |1-5|^2}$$

any number squared is positive

$$= \sqrt{(2-4)^2 + (1-5)^2}$$

$$= \sqrt{4+16} = \sqrt{20}$$

Idea: For $a,b \in \mathbb{R}^2$

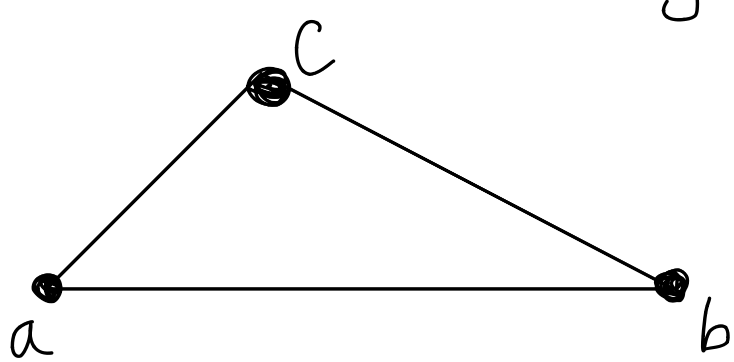
$a = (x_1, y_1) \quad b = (x_2, y_2)$

$$d(a,b) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Is $d((2,1),(4,5)) = d((4,5),(2,1))$?

$$d((4,5),(2,1)) = \sqrt{(4-2)^2 + (5-1)^2}$$
$$= \sqrt{4 + 16}$$
$$= \sqrt{20} \quad \text{yes!}$$

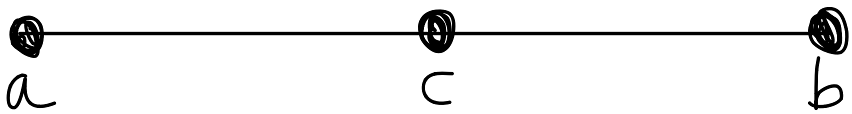
One more thing...
when we think of distance, there's also a notion of efficiency.



The path directly from a to b covers less distance than the path from a to b with a stop in between at c. That is...

$$\boxed{\text{Goal 3: } d(a,b) < d(a,c) + d(c,b)}$$

What if c is on the way?



Then $d(a,b) = d(a,c) + d(c,b)$

$$\boxed{\text{Goal 3 (modified): } d(a,b) \le d(a,c) + d(c,b)}$$

Let X be a set (in our examples, $X = \mathbb{R}$ or $X = \mathbb{R}^2$). A **metric** on X is a function $d: X \times X \to \mathbb{R}$ such that for all $a, b, c \in X$

1. $d(a,b) > 0$ if $a \neq b$
   $d(a,b) = 0$ if $a = b$
2. $d(a,b) = d(b,a)$
3. $d(a,b) \le d(a,c) + d(c,b)$
   triangle inequality

standard/usual/Euclidean metric

For $a, b \in \mathbb{R}$, $d(a,b) = |a-b|$

For $a, b \in \mathbb{R}^2$

$a = (x_1, y_1) \quad b = (x_2, y_2)$

$$d(a,b) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$

A metric space is a pair (X, d) where X is a set and d is a metric on X.

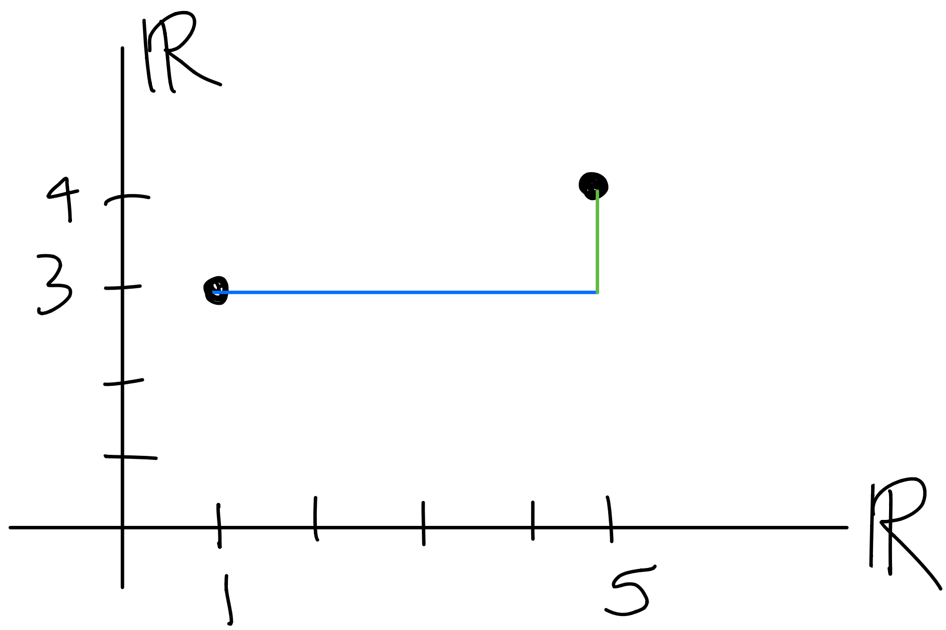
Example: $(\mathbb{R}^2, d)$ where d is the taxicab metric

For $a, b \in \mathbb{R}^2$

$a = (x_1, y_1) \quad b = (x_2, y_2)$

$$d(a,b) = |\underline{x_1 - x_2}| + |\underline{y_1 - y_2}|$$

$a = (1,3) \qquad b = (5,4)$



Proofs and Counterexamples

$\Rightarrow$ means "implies"

$\blacksquare$ means argument is complete

[standard/usual/Euclidean metric
For $a, b \in \mathbb{R}$, $d(a,b) = |a-b|$]

1. $d(a,b) > 0$ if $a \neq b$

$$a \neq b \Rightarrow a-b \neq 0$$

$$\Rightarrow |a-b| \neq 0$$

$$\Rightarrow |a-b| > 0 \quad \text{definition of abs. val.}$$

$$\Rightarrow d(a,b) > 0$$

$d(a,b) = 0$ if $a = b$

$$a = b \Rightarrow a-b = 0$$

$$\Rightarrow |a-b| = 0$$

$$\Rightarrow d(a,b) = 0$$

2. $d(a,b) = d(b,a)$

$$\begin{aligned} d(a,b) &= |a-b| \\ &= |-(a-b)| \\ &= |b-a| \\ &= d(b,a) \end{aligned}$$

3. $d(a,b) \leq d(a,c) + d(c,b)$

This follows from a more general result called The Triangle Inequality

Theorem (the Triangle Inequality)

If $x,y \in \mathbb{R}$, then $|x+y| \leq |x| + |y|$

Proof:

$$\begin{aligned} |x+y|^2 &= (x+y)^2 \\ &= x^2 + 2xy + y^2 \\ &\leq x^2 + 2|xy| + y^2 \end{aligned}$$

$$= x^2 + 2|x||y| + y^2$$

$\sqrt{a^2} = \pm a$ ie $\sqrt{a^2} = |a|$

$$\Rightarrow \sqrt{(xy)^2} = \sqrt{x^2 y^2}$$
$$= \sqrt{x^2}\sqrt{y^2}$$
$$= |x||y|$$

$$= |x|^2 + 2|x||y| + |y|^2$$

$$= (|x| + |y|)^2$$

$$\Rightarrow |x+y|^2 \leq (|x| + |y|)^2$$

$$\Rightarrow |x+y| \leq |x| + |y|$$

Technically $|x+y| \leq |x|+|y|$

and $|x+y| \geq -|x| - |y|$

but $|x+y| \geq -|x| - |y| \Rightarrow \underbrace{-|x+y| \leq |x|+|y|}_{\text{always true}}$

Now,

$$
\begin{aligned}
d(a,b) &= |a-b| \\
&= |a+0-b| \\
&= |\underbrace{a-c}_{x}+\underbrace{c-b}_{y}| \\
&\leq |a-c|+|c-b| \quad \text{Triangle Ineq in } \mathbb{R} \\
&= d(a,c)+d(c,b)
\end{aligned}
$$

Therefore, d is a metric on $\mathbb{R}$. $\blacksquare$

standard/usual/Euclidean metric

For $a, b \in \mathbb{R}^2$

$a = (x_1, y_1) \quad b = (x_2, y_2)$

$$d(a,b) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

1. $d(a,b) > 0$ if $a \neq b$

$a \neq b \Rightarrow (x_1, y_1) \neq (x_2, y_2)$

$\Rightarrow x_1 \neq x_2$ or $y_1 \neq y_2$

$\Rightarrow x_1 - x_2 \neq 0$ or $y_1 - y_2 \neq 0$

$\Rightarrow (x_1 - x_2)^2 \neq 0$ or $(y_1 - y_2)^2 \neq 0$

$\Rightarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 \neq 0$

$\Rightarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 > 0$

$\Rightarrow \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} > 0$

$\Rightarrow d(a,b) > 0$

$d(a,b)=0$ if $a=b$

$$a=b \Rightarrow (x_1,y_1)=(x_2,y_2)$$
$$\Rightarrow x_1=x_2 \text{ and } y_1=y_2$$
$$\Rightarrow x_1-x_2=0 \text{ and } y_1-y_2=0$$
$$\Rightarrow (x_1-x_2)^2=0 \text{ and } (y_1-y_2)^2=0$$
$$\Rightarrow (x_1-x_2)^2+(y_1-y_2)^2=0$$
$$\Rightarrow \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=0$$
$$\Rightarrow d(a,b)=0$$

2. $d(a,b)=d(b,a)$

$$d(a,b)=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$
$$=\sqrt{(-(x_1-x_2))^2+(-(y_1-y_2))^2}$$
$$=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$=d(b,a)$$

3. $d(a,b) \leq d(a,c) + d(c,b)$

This follows from a more general result called the Cauchy Schwarz Inequality.

Recall how to compute a dot product.

$$a = (x_1, y_1) \in \mathbb{R}^2$$
$$b = (x_2, y_2) \in \mathbb{R}^2$$

$$a \cdot b = x_1 x_2 + y_1 y_2$$

$$\text{Let } \overset{\text{norm}}{\|a\|} = d(a,0)$$
$$= d((x_1, y_1), (0,0))$$
$$= \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2}$$
$$= \sqrt{(x_1)^2 + (y_1)^2}$$
$$= \sqrt{a \cdot a}$$

and $\|b\| = d(b,0)$

$$= \sqrt{(x_2)^2 + (y_2)^2}$$

$$= \sqrt{b \cdot b}$$

Theorem (Cauchy Schwarz)

$$|a \cdot b| \leq \|a\| \cdot \|b\|$$

Note 1: $d(a,b) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$$= d((x_1 - x_2, y_1 - y_2), (0,0))$$

$$= d(a - b, 0)$$

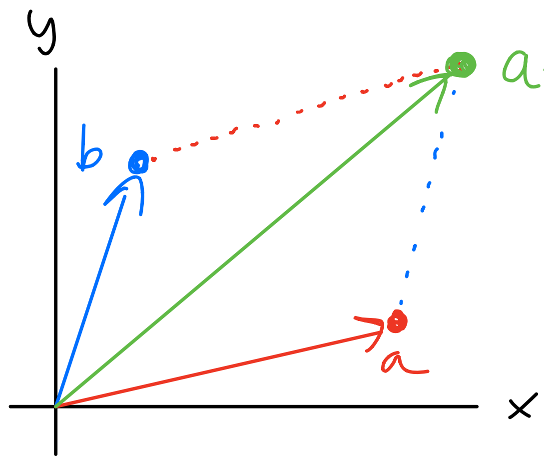
$$= \|a - b\|$$

Note 2: $\|a\|^2 = a \cdot a$

$$
\begin{aligned}
\|a+b\|^2 &= (a+b)\cdot(a+b) \\
&= (x_1+x_2, y_1+y_2)\cdot(x_1+x_2, y_1+y_2) \\
&= (x_1+x_2)^2 + (y_1+y_2)^2 \\
&= (x_1)^2 + 2x_1x_2 + (x_2)^2 + (y_1)^2 + 2y_1y_2 + (y_2)^2 \\
&= \underline{(x_1)^2+(y_1)^2} + 2\underline{(x_1x_2+y_1y_2)} + \underline{(x_2)^2+(y_2)^2} \\
&= a\cdot a + 2(a\cdot b) + b\cdot b \\
&= \|a\|^2 + 2(a\cdot b) + \|b\|^2 \\
&\leq \|a\|^2 + 2|a\cdot b| + \|b\|^2 \\
&\leq \|a\|^2 + 2\|a\|\|b\| + \|b\|^2 \quad \text{by CS} \\
&= (\|a\|+\|b\|)^2
\end{aligned}
$$

$$\Rightarrow \|a+b\|^2 \le (\|a\|+\|b\|)^2$$

$$\Rightarrow \boxed{\|a+b\| \le \|a\|+\|b\|}$$



NOW,

$$\begin{aligned} d(a,b) &= \|a-b\| \\ &= \|a+0-b\| \\ &= \|a-c+c-b\| \\ &\le \|a-c\|+\|c-b\| \end{aligned}$$

$$= d(a, c) + d(c, b)$$

Therefore, d is a metric on $\mathbb{R}^2$ $\blacksquare$

taxicab metric

For $a,b \in \mathbb{R}^2$

$a = (x_1, y_1) \quad b = (x_2, y_2)$

$$d(a,b) = |x_1 - x_2| + |y_1 - y_2|$$

1. $d(a,b) > 0$ if $a \neq b$

$a \neq b \Rightarrow (x_1, y_1) \neq (x_2, y_2)$

$\Rightarrow x_1 \neq x_2$ or $y_1 \neq y_2$

$\Rightarrow x_1 - x_2 \neq 0$ or $y_1 - y_2 \neq 0$

$\Rightarrow |x_1 - x_2| \neq 0$ or $|y_1 - y_2| \neq 0$

$\Rightarrow |x_1 - x_2| > 0$ or $|y_1 - y_2| > 0$

$\Rightarrow |x_1 - x_2| + |y_1 - y_2| > 0$

$\Rightarrow d(a,b) > 0$

$d(a,b)=0$ if $a=b$

$a=b \Rightarrow (x_1,y_1)=(x_2,y_2)$

$\Rightarrow x_1=x_2$ and $y_1=y_2$

$\Rightarrow x_1-x_2=0$ and $y_1-y_2=0$

$\Rightarrow |x_1-x_2|=0$ and $|y_1-y_2|=0$

$\Rightarrow |x_1-x_2|+|y_1-y_2|=0$

$\Rightarrow d(a,b)=0$

2. $d(a,b)=d(b,a)$

$d(a,b)=|x_1-x_2|+|y_1-y_2|$

$=|-(x_1-x_2)|+|-(y_1-y_2)|$

$=|x_2-x_1|+|y_2-y_1|$

$=d(b,a)$

3. $d(a,b) \leq d(a,c) + d(c,b)$ $\quad c = (x_3, y_3)$

$$d(a,b) = |x_1 - x_2| + |y_1 - y_2|$$

$$= |x_1 + 0 - x_2| + |y_1 + 0 - y_2|$$

$$= |x_1 - x_3 + x_3 - x_2| + |y_1 - y_3 + y_3 - y_2|$$

$$\leq |x_1 - x_3| + |x_3 - x_2| + |y_1 - y_3| + |y_3 - y_2|$$

Triangle Inequality in $\mathbb{R}$

$$= |x_1 - x_3| + |y_1 - y_3| + |x_3 - x_2| + |y_3 - y_2|$$

$$= d(a,c) + d(c,b)$$

Therefore, d is a metric on $\mathbb{R}^2$. ∎