Ordered Sets and Bounds 1

Determine whether or not the following relations are orders.

1. $(E, \ll)$ where $(S, <)$ is an ordered set, E is a subset of S, and for $x, y \in E$, $x \ll y$ if $x < y$.

2. $(\mathbb{R}, <_f)$ where $f\colon \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ and we say $x <_f y$ if $f(x) < f(y)$.

3. $(\mathbb{R}, <_g)$ where $g\colon \mathbb{R} \to \mathbb{R}$ with $g(x) = x^3$ and we say $x <_g y$ if $g(x) < g(y)$.

4. $(S, <)$ where S is the set of organisms in an ecosystem and $x < y$ if y eats x.

5. $(S, <)$ where S is the set of people in a family tree and $x < y$ if x is a descendant of y.

6. $(S_1 \cup S_2, <)$ where $(S_1, <_1)$ and $(S_2, <_2)$ are disjoint ordered sets and $x < y$ if either $x, y \in S_1$ with $x <_1 y$; $x, y \in S_2$ with $x <_2 y$; or $x \in S_1$ and $y \in S_2$.

7. $(\mathbb{R}^2, <)$ where $(x_1, y_1) < (x_2, y_2)$ if $x_1 < x_2$ and $y_1 < y_2$.

Determine if the following subsets of ordered sets are bounded above and/or below, and if so, if they have an infimum and/or supremum in that set.

8. $\{(-1)^n n \mid n \text{ positive integer}\}$ as a subset of $\mathbb{R}$

9. $\operatorname{Im} f$ as a subset of $\mathbb{R}$, where $f\colon (\frac{\pi}{3}, \frac{5\pi}{6}) \to \mathbb{R}$ defined by $f(x) = \sin x$.

10. $\{2, 2.2, 2.22, 2.222, \ldots\}$ as a subset of $\mathbb{R}$.

11. $\{2, 2.2, 2.22, 2.222, \ldots\}$ as a subset of $\mathbb{Q}$.

Ordered Sets and Bounds 2

1. We stated in the notes today that $\sqrt{2}$ is irrational. Prove it!

   Hint: Suppose by contradiction that $\sqrt{2}$ is rational. Then $\sqrt{2}$ can be written as $\frac{p}{q}$, where p, q are integers that are relatively prime. Proceed from here to find a contradiction.

2. Does $\mathbb{R}^2$ with the dictionary order have the least upper bound property? Prove or provide a counterexample.

3. Let $A \subseteq \mathbb{R}$ be nonempty and bounded below. Define $-A = \{-x \mid x \in A\}$. Prove that $-A$ is bounded above and $\inf A = -\sup(-A)$.

4. Let S be an ordered set that has the greatest lower bound property. Prove that S has the least upper bound property.

5. Let x be a positive real number and n a positive integer.

   (a) Let $A = \{t \in \mathbb{R} \mid t > 0 \text{ and } t^n < x\}$. Show A has a supremum in $\mathbb{R}$.
   Hint: Show $\frac{x}{1+x} \in A$ and $1 + x$ is an upper bound of A.

   (b) Prove that $b^n - a^n < (b - a)nb^{n-1}$ whenever $0 < a < b$.
   Hint: Factor $b^n - a^n$.

   (c) In the rest of this problem, let $y = \sup A$. Prove that $y^n \nless x$.
   Hint: Assume that $y^n < x$. Pick $h \in \mathbb{R}$ with $0 < h < \min\left\{1, \frac{x-y^n}{n(y+1)^{n-1}}\right\}$. Show that $y + h \in A$ to get a contradiction.

   (d) Prove that $y^n \ngtr x$ and conclude that there exists a positive n^{th} root of x.
   Hint: Assume $y^n > x$. Let $k = \frac{y^n-x}{ny^{n-1}}$ and show $y - k$ is an upper bound of A to get a contradiction.

   (e) Prove there is exactly one positive real n^{th} root of x.