Notation:

$\Rightarrow$ means implies

Example: it's raining $\Rightarrow$ the ground is wet

What about the converse?

the ground is wet $\Rightarrow$ it's raining?

That's not always true! Maybe the ground is wet because the sprinklers are on.

If both a statement and its converse are true, we use the symbol $\Leftrightarrow$ which means if and only if

Example: you are a parent $\Rightarrow$ you have a child

you have a child $\Rightarrow$ you are a parent

Thus, you are a parent $\Leftrightarrow$ you have a child.

Math example 1:

$f: \mathbb{R} \to \mathbb{R}$ is differentiable $\Rightarrow$ f is continuous

$f:\mathbb{R}\to\mathbb{R}$ is continuous $\not\Rightarrow$ f is differentiable

(think about $f(x)=|x|$)

Math example 2:

$2x+3=4 \Rightarrow 2x=1$ (subtract 3 from both sides)

$2x=1 \Rightarrow 2x+3=4$ (add 3 to both sides)

Thus, $2x+3=4 \Leftrightarrow 2x=1$

Other symbols:

$\forall$ means "for all/any/every"

Example: $x^2 \geq 0 \ \forall x \in \mathbb{R}$

$\exists$ means "there exists"

Example: $\exists$ a solution to the equation $3x+5=0$

st means "such that"

Sequences and Subsequences

Topic 1: Convergence

natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$

Let (X, d) be a metric space. A sequence in X is the list of outputs of a function

$$f: \mathbb{N} \to X, \quad n \mapsto a_n$$

and is denoted $\{a_n\}$.

Example 1: $f: \mathbb{N} \to \mathbb{R}$, $n \mapsto \frac{1}{n}$ or $a_n = \frac{1}{n}$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

You may recognize this as a harmonic sequence.

Example 2: $f: \mathbb{N} \to \mathbb{R}$, $n \mapsto 5n+1$ or $a_n = 5n+1$

$6, 11, 16, 21, 26, 31, \ldots$

Example 3: $a_n = \frac{n+1}{n+2}$

$\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots$

Example 4: Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, 21, 34

$a_1 = 1$

$a_2 = 1$

$a_n = a_{n-2} + a_{n-1} \quad (n \geq 3)$

This sequence is called recursive.

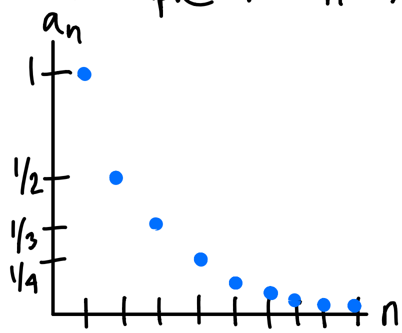
Example 5: $a_n = -1$ if n is odd
$a_n = 1$ if n is even

$-1, 1, -1, 1, -1, 1, -1, 1, \ldots$

converge: to move toward one point

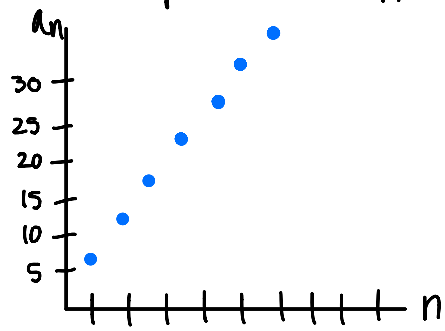
Do these sequences converge?

Example 1: $a_n = 1/n$



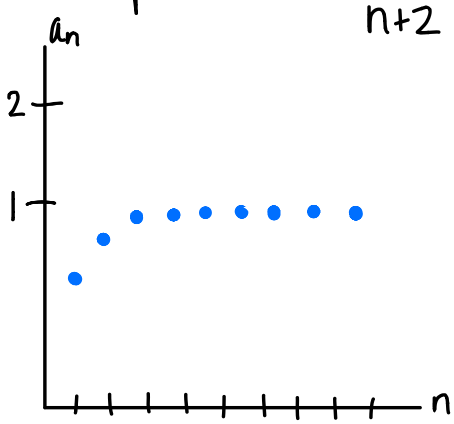
converging to 0

Example 2: $a_n = 5n+1$



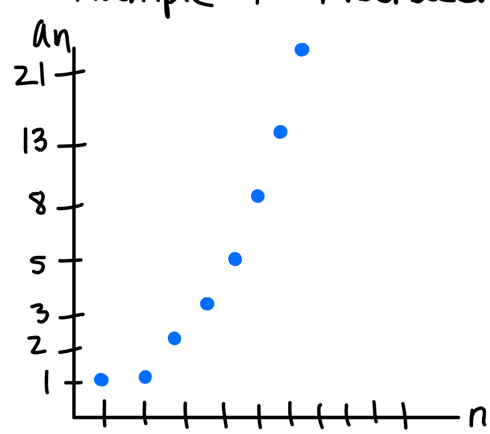
doesn't converge
(increases to infinity)

Example 3: $a_n = \frac{n+1}{n+2}$



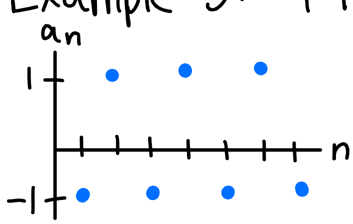
converging to 1

Example 4: Fibonacci



doesn't converge
(increases to infinity)

Example 5: -1 if n is odd, 1 if n is even



doesn't converge
doesn't approach anything, not even ∞

You saw this in calculus written as

$$\lim_{x\to\infty} \frac{1}{x} = 0 \qquad \lim_{x\to\infty}(5x+1) = \infty \qquad \lim_{x\to\infty} \frac{x+1}{x+2} = 1$$

Mathematically, what does it mean when we say $a_n = 1/n$ converges to zero?

Notation: $a_n \longrightarrow 0$ or $\lim_{n\to\infty} a_n = 0$

It means that $\overset{n>}{\underline{\text{past}}}$ a $\underline{\text{certain}\overset{N}{\ }\text{point}}$ in the sequence, all the numbers in the sequence are <u>really close to 0</u>.

There is an $N \in \mathbb{N}$ such that

$$\underline{n > N} \Rightarrow \underline{d(a_n, 0) \approx 0}$$

$\uparrow$

How close to zero?

<u>Arbitrarily</u> close.

$\downarrow$

$d(a_n, 0) < \varepsilon$ for any $\varepsilon \in \mathbb{R}$ st $\varepsilon > 0$

A sequence $\{a_n\}$ in a metric space (X,d) converges to $a \in X$ if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ such that $n > N \Rightarrow d(a_n, a) < \varepsilon$. We say a is the limit of $\{a_n\}$.

If $\{a_n\}$ does not converge, then $\{a_n\}$ diverges.

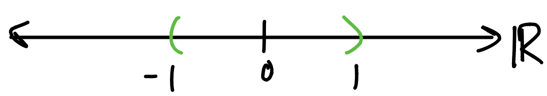
Topic 2: Boundedness

Let $p \in X$ and $r > 0$.

The ball in X centered at p with radius r is the set $B(p,r) = \{x \in X \mid d(x,p) < r\}$.

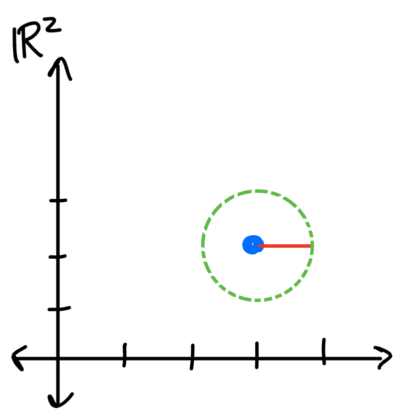
Example 1: $X = \mathbb{R}$ and d is the usual metric

$B(0,1)$



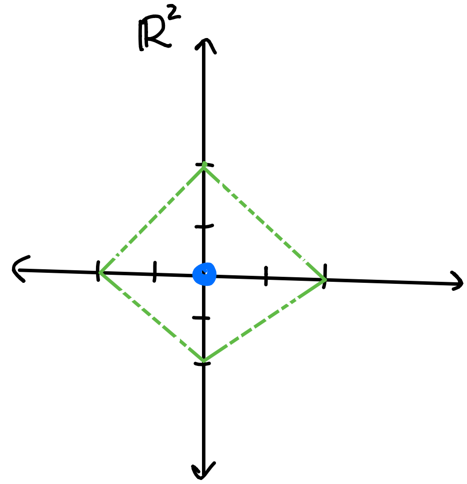
Example 2: $X = \mathbb{R}^2$ and d is the usual metric

$B(\underline{(3,2)}, \underline{1})$



Example 3: $X = \mathbb{R}^2$ and d is the taxicab metric

$B((0,0), 2)$



We saw yesterday what it means for a subset of an ordered set $(S, <)$ to be bounded. Now...

Let (X, d) be a metric space and $E \subseteq X$.

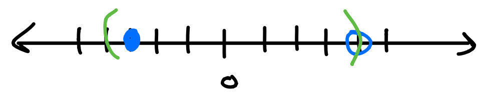
E is bounded if $\exists M \in \mathbb{R}$ and $q \in X$ st

$d(p, q) < M \quad \forall p \in E$.

In other words, $E \subseteq B(q, M)$.

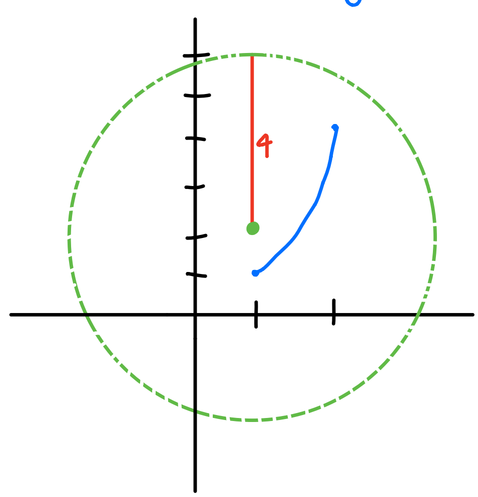
Example 1: $X = \mathbb{R}$ and d is the usual metric

$E = [-3, 4) \subseteq B(0, 4)$



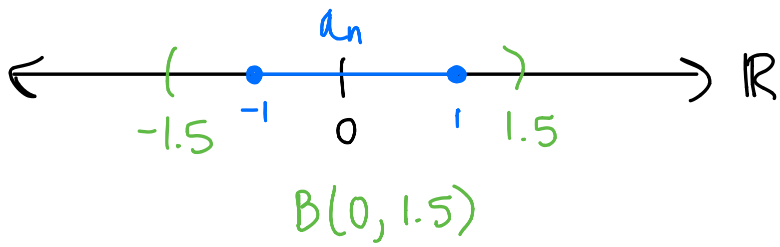
Example 2: $X = \mathbb{R}^2$ and d is the usual metric

$E = \{ (x,y) \in \mathbb{R}^2 \mid 1 \leq x \leq 2 \text{ and } y = x^2 \} \subseteq B((1,2), 4)$

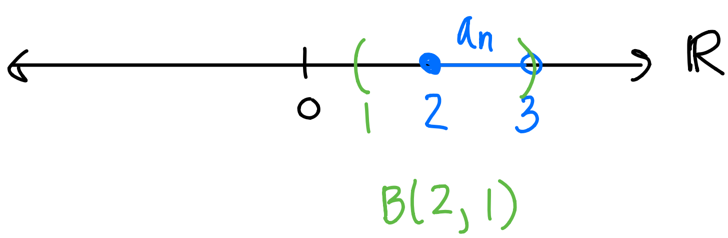


A sequence in a metric space (X, d) can be thought of as a subset of X. Thus, a sequence $\{a_n\}$ in X is bounded if $\{a_n\} \subseteq X$ is bounded as defined above.

Example 1: $a_n = \sin(n)$ is bounded in $\mathbb{R}$



Example 2: $a_n = -1/n^2 + 3$ is bounded in $\mathbb{R}$



Example 3: $a_n = 2^n$ is not bounded

(X,d) is a metric space.

Theorem The limit of a convergent sequence is unique.

Proof: (by contradiction)

Let $\{a_n\} \subseteq X$ be a sequence that converges to both $x_1 \in X$ and $x_2 \in X$ where $x_1 \neq x_2$.

$x_1 \neq x_2 \Rightarrow d(x_1, x_2) > 0$

Let $\varepsilon = \dfrac{d(x_1, x_2)}{2}$

Since $a_n \longrightarrow x_1$ $\exists N_1 \in \mathbb{N}$ st

$n > N_1 \Rightarrow d(a_n, x_1) < \varepsilon = \dfrac{d(x_1, x_2)}{2}$

and since $a_n \longrightarrow x_2$ $\exists N_2 \in \mathbb{N}$ st

$n > N_2 \Rightarrow d(a_n, x_2) < \varepsilon = \dfrac{d(x_1, x_2)}{2}$

Then $d(x_1, x_2) \leq d(x_1, a_n) + d(a_n, x_2) \quad \forall n \in \mathbb{N}$

$= d(a_n, x_1) + d(a_n, x_2)$

$$< \frac{d(x_1,x_2)}{2} + \frac{d(x_1,x_2)}{2} \quad \text{when } n > N_1 \text{ and } n > N_2$$

$$= d(x_1,x_2)$$

So $d(x_1,x_2) < d(x_1,x_2) \xrightarrow{\text{contradiction}}\leftarrow$ A number is not less than itself.

Thus, no such distinct limits a_1, a_2 exist.

Therefore, the limit of a convergent sequence is unique. $\blacksquare$

Theorem Every convergent sequence is bounded.

Proof: Let $\{a_n\} \subseteq X$ converge to $a \in X$. Let $\varepsilon = 1$.

Since $a_n \to a$ $\exists N \in \mathbb{N}$ st $n > N \Rightarrow d(a_n, a) < \varepsilon = 1$.

Consider the first N terms $a_1, a_2, \ldots, a_N$ in the sequence.

Let $M = \max\{ d(a_1,a), d(a_2,a), \ldots, d(a_N,a), \underbrace{1}_{M \geq 1} \}$.

Then $d(a_n, a) \leq M$ when $1 \leq n \leq N$ and

$d(a_n, a) < 1 \leq M$ when $n > N$, so $d(a_n, a) \leq M$ $\forall n \in \mathbb{N}$.

Thus, $\{a_n\} \subseteq B(a, M+1)$.

Therefore, $\{a_n\}$ is bounded. ■

Converse: If $\{a_n\}$ is bounded, then $\{a_n\}$ converges.

This is false!

(worksheet 2)

A sequence $\{a_n\}$ is increasing if

$a_{n+1} \geq a_n$ $\forall n \in \mathbb{N}$.

Example: $1, 3, 5, 7, 9, 11, \ldots, a_n = 2n-1$

A sequence $\{a_n\}$ is decreasing if $a_{n+1} \leq a_n$

$\forall n \in \mathbb{N}$.

Example: $-2, -4, -6, -8, -10, \ldots, a_n = -2n$

A sequence $\{a_n\}$ is monotonic if $\{a_n\}$ is increasing or $\{a_n\}$ is decreasing.

Theorem If $\{a_n\}$ is bounded <u>and monotonic</u>, then $\{a_n\}$ converges.

Proof: worksheet 2

Subsequences

Example 1:

sequence: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

subsequence: $1, 2, 5, 13, 34, \ldots$

Example 2:

sequence: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, ...

subsequence: 8, 16, 18, 20, 30, 40, 52, 58, ...

<u>Rule:</u> we can't choose a term more than once.

Example:

sequence: 1, 2, 4, 7, 11, 16, 22, 5, 5, 5, 27, ...

not a subsequence: 4, 11, 11, 22, 5, 5, 27, ...

Notation:

Let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of natural numbers (no repetition allowed)

The sequence $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$.

Example : $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, \ldots$

$a_1, a_3, a_5, a_7, a_9, \ldots$

$\{n_k\}_{k=1}^{\infty} = \{1, 3, 5, 7, 9, \ldots\}$

Theorem If $a_n \to a$, then every subsequence of $\{a_n\}$ converges to a.

Proof: Let $\{a_n\}$ be a sequence st $a_n \to a$.

Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$.

$a_n \to a \Rightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}$ st $n > N \Rightarrow$

$d(a_n, a) < \varepsilon$.

Note: $n_k \geq k \quad \forall k \in \mathbb{N}$

Example : $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, \ldots$

$a_1, a_3, a_5, a_7, a_9, \ldots$

$n_1 = 1$ $\quad n_2 = 3$ $\quad n_3 = 5$ $\quad n_4 = 7$ $\quad n_5 = 9$

Let $K \geq N$. By the note, $n_K \geq K$.

Thus, $n_K \geq K \geq N \implies n_K \geq N$

$$\implies d(a_{n_K}, a) < \varepsilon$$

Therefore, $a_{n_K} \to a$. $\blacksquare$

Proving a specific sequence converges

Claim: $a_n = 1/n \longrightarrow 0$

Proof: Let $\varepsilon > 0$.

We must find some $N \in \mathbb{N}$ so that $d(\frac{1}{n}, 0) < \varepsilon$ whenever $n > N$.

$$
\begin{aligned}
d\left(\tfrac{1}{n}, 0\right) &= \left|\tfrac{1}{n} - 0\right| \\
&= \left|\tfrac{1}{n}\right| \\
&= \tfrac{1}{n} \quad \text{since } n \in \mathbb{N} \Rightarrow n > 0 \\
&< \varepsilon \quad \text{when } n > \tfrac{1}{\varepsilon}
\end{aligned}
$$

side work: $\frac{1}{n} < \varepsilon$

$\Leftrightarrow 1 < \varepsilon n$

$\Leftrightarrow \frac{1}{\varepsilon} < n$

$\Leftrightarrow n > \frac{1}{\varepsilon}$

Therefore, $\frac{1}{n} \longrightarrow 0$ ∎

Claim: $a_n = \frac{n+1}{n+2} \longrightarrow 1$

Proof: Let $\varepsilon > 0$

$$d\left(\frac{n+1}{n+2}, 1\right) = \left|\frac{n+1}{n+2} - 1\right|$$

$$= \left|\frac{n+1-(n+2)}{n+2}\right|$$

$$= \left|\frac{n+1-n-2}{n+2}\right|$$

$$= \left|\frac{-1}{n+2}\right|$$

$$= \frac{1}{n+2} \qquad \text{(remember } n>0\text{)}$$

$$< \varepsilon \quad \text{when} \quad n > \frac{1}{\varepsilon} - 2$$

side work: $\frac{1}{n+2} < \varepsilon$

$\Leftrightarrow 1 < \varepsilon(n+2)$

$\Leftrightarrow \frac{1}{\varepsilon} < n+2$

$\Leftrightarrow \frac{1}{\varepsilon} - 2 < n$

$\Leftrightarrow n > \frac{1}{\varepsilon} - 2$

Therefore, $\frac{n+1}{n+2} \longrightarrow 1$ $\blacksquare$

More examples

1. Prove $a_n = \dfrac{\cos(n)}{n^2+17} \longrightarrow 0$

Proof: Let $\varepsilon > 0$.

$$d\left(\frac{\cos(n)}{n^2+17}, 0\right) = \left|\frac{\cos(n)}{n^2+17} - 0\right|$$

$$= \left|\frac{\cos(n)}{n^2+17}\right|$$

$$= \frac{|\cos(n)|}{|n^2+17|}$$

$$= \frac{|\cos(n)|}{n^2+17}$$

$$\leq \frac{1}{n^2+17} \qquad -1 \leq \cos(n) \leq 1 \implies |\cos(n)| \leq 1$$

$$< \frac{1}{n^2}$$

$$< \varepsilon \text{ if } n > \sqrt{\frac{1}{\varepsilon}}$$

side work: $\frac{1}{n^2} < \varepsilon$

$\iff 1 < \varepsilon n^2$

$\iff \frac{1}{\varepsilon} < n^2$

$\iff n^2 > \frac{1}{\varepsilon}$

$\iff n > \sqrt{\frac{1}{\varepsilon}}$

Therefore, $\dfrac{\cos(n)}{n^2+17} \longrightarrow 0$ ∎

2. $a_n = \dfrac{(1+4n)^2}{1-3n-6n^2} \longrightarrow -\dfrac{8}{3}$

Proof: Let $\varepsilon > 0$.

$$d\left(\frac{(1+4n)^2}{1-3n-6n^2}, -\frac{8}{3}\right) = \left|\frac{(1+4n)^2}{1-3n-6n^2} - \left(-\frac{8}{3}\right)\right|$$

$$= \left|\frac{1+8n+16n^2}{1-3n-6n^2} + \frac{8}{3}\right|$$

$$= \left|\frac{3(1+8n+16n^2)+8(1-3n-6n^2)}{3(1-3n-6n^2)}\right|$$

$$= \left|\frac{3+24n+48n^2+8-24n-48n^2}{3(1-3n-6n^2)}\right|$$

$$= \left|\frac{11}{3(1-3n-6n^2)}\right|$$

$$= \frac{11}{3|1-3n-6n^2|}$$

$$< \frac{11}{3|-3n-6n^2|}$$

$$= \frac{11}{3|-3(n+2n^2)|}$$

$$= \frac{11}{9|n+2n^2|}$$

$$= \frac{11}{9(n+2n^2)} \qquad n>0 \Rightarrow n+2n^2>0$$

$$< \frac{11}{9n}$$

$$< \varepsilon \quad \text{when } n > \frac{11}{9\varepsilon}$$

side work: $\frac{11}{9n} < \varepsilon$

$\iff 11 < 9\varepsilon n$

$\iff \frac{11}{9\varepsilon} < n$

$\iff n > \frac{11}{9\varepsilon}$

Therefore, $\frac{(1+4n)^2}{1-3n-6n^2} \longrightarrow \frac{-8}{3}$ $\blacksquare$