Solutions

Sequences and Subsequences 1

1. $\lim\limits_{n\to\infty} \frac{16}{n^3} = 0$ (a_n converges to zero)

2. $\lim\limits_{n\to\infty} \frac{80}{\sqrt{5n}} = 0$ (a_n converges to zero)

3. $\lim\limits_{n\to\infty} \cos(n)$ doesn't exist (a_n doesn't converge)

4. $\lim\limits_{n\to\infty} \left(1 - \frac{(-1)^n}{n}\right) = 1$ (a_n converges to 1)

5. $\lim\limits_{n\to\infty} \frac{9-7n}{8+13n} = \frac{-7}{13}$ (a_n converges to $^{-7}/_{13}$)

6. $\lim\limits_{n\to\infty} (-n) = -\infty$ (a_n doesn't converge)

7. $\lim\limits_{n\to\infty} \frac{(1+2n)^2}{5+3n+3n^2} = \frac{4}{3}$ (a_n converges to $\frac{4}{3}$)

8. yes $B(0, 17) = (-17, 17)$

9. yes $B(0, 36) = (-36, 36)$

10. yes $B(0, 2) = (-2, 2)$

11. yes $B(0, 3) = (-3, 3)$

12. yes $B(0, 2) = (-2, 2)$

13. no

14. yes $B(0, 4) = (-4, 4)$

Solutions

Sequences and Subsequences 2

1. $a_n = \frac{16}{n^3} \longrightarrow 0$

Let $\varepsilon > 0$.

$$d\left(\frac{16}{n^3}, 0\right) = \left|\frac{16}{n^3} - 0\right|$$
$$= \left|\frac{16}{n^3}\right|$$
$$= \frac{16}{n^3} \quad \text{since } n > 0$$
$$< \varepsilon \quad \text{when } n > \sqrt[3]{\frac{16}{\varepsilon}}$$

side work: $\frac{16}{n^3} < \varepsilon$

$$\iff 16 < \varepsilon n^3$$
$$\iff \frac{16}{\varepsilon} < n^3$$
$$\iff n^3 > \frac{16}{\varepsilon}$$
$$\iff n > \sqrt[3]{\frac{16}{\varepsilon}}$$

Therefore, $\frac{16}{n^3} \longrightarrow 0$ ∎

2. $a_n = \frac{80}{\sqrt{5n}} \longrightarrow 0$

Let $\varepsilon > 0$.

$$d\left(\frac{80}{\sqrt{5n}}, 0\right) = \left|\frac{80}{\sqrt{5n}} - 0\right|$$
$$= \left|\frac{80}{\sqrt{5n}}\right|$$
$$= \frac{80}{\sqrt{5n}}$$
$$< \frac{80}{\sqrt{n}}$$
$$< \varepsilon \quad \text{when } n > \frac{6400}{\varepsilon^2}$$

side work: $\frac{80}{\sqrt{n}} < \varepsilon$

$\Longleftrightarrow 80 < \varepsilon\sqrt{n}$

$\Longleftrightarrow \frac{80}{\varepsilon} < \sqrt{n}$

$\Longleftrightarrow \sqrt{n} > \frac{80}{\varepsilon}$

$\Longleftrightarrow n > \frac{6400}{\varepsilon^2}$

Therefore, $\frac{80}{\sqrt{5n}} \longrightarrow 0$ ∎

3. $a_n = 1 - \frac{(-1)^n}{n} \longrightarrow 1$

Let $\varepsilon > 0$.

$$d\left(1 - \frac{(-1)^n}{n}, 1\right) = \left| 1 - \frac{(-1)^n}{n} - 1 \right|$$
$$= \left| -\frac{(-1)^n}{n} \right|$$
$$= \frac{1}{n}$$
$$< \varepsilon \quad \text{if} \quad n > \frac{1}{\varepsilon}$$

side work: $\frac{1}{n} < \varepsilon$

$\Leftrightarrow 1 < \varepsilon n$

$\Leftrightarrow \frac{1}{\varepsilon} < n$

$\Leftrightarrow n > \frac{1}{\varepsilon}$

Therefore, $1 - \frac{(-1)^n}{n} \longrightarrow 1$ $\blacksquare$

4. $a_n = \dfrac{9-7n}{8+13n} \longrightarrow \dfrac{-7}{13}$

Let $\varepsilon > 0$.

$$
\begin{aligned}
d\left(\frac{9-7n}{8+13n}, \frac{-7}{13}\right) &= \left|\frac{9-7n}{8+13n} - \left(\frac{-7}{13}\right)\right| \\
&= \left|\frac{9-7n}{8+13n} + \frac{7}{13}\right| \\
&= \left|\frac{13(9-7n)+7(8+13n)}{13(8+13n)}\right| \\
&= \left|\frac{117-91n+104+91n}{13(8+13n)}\right| \\
&= \left|\frac{221}{13(8+13n)}\right| \\
&= \frac{221}{13(8+13n)} \qquad \text{since } n>0 \\
&< \frac{221}{8+13n} \\
&< \frac{221}{13n} \\
&< \frac{221}{n} \\
&< \varepsilon \quad \text{when } n > \frac{221}{\varepsilon}
\end{aligned}
$$

side work: $\dfrac{221}{n} < \varepsilon$

$\Longleftrightarrow n > \dfrac{221}{\varepsilon}$

Therefore, $\dfrac{9-7n}{8+13n} \longrightarrow \dfrac{-7}{13}$ ∎

5. $a_n = \dfrac{(1+2n)^2}{5+3n+3n^2} \longrightarrow \dfrac{4}{3}$

Let $\varepsilon > 0$.

$$d\left(\frac{(1+2n)^2}{5+3n+3n^2}, \frac{4}{3}\right) = \left|\frac{(1+2n)^2}{5+3n+3n^2} - \frac{4}{3}\right|$$

$$= \left|\frac{3(1+4n+4n^2) - 4(5+3n+3n^2)}{3(5+3n+3n^2)}\right|$$

$$= \left|\frac{3+12n+12n^2 - 20 - 12n - 12n^2}{3(5+3n+3n^2)}\right|$$

$$= \left|\frac{-17}{3(5+3n+3n^2)}\right|$$

$$= \frac{17}{3(5+3n+3n^2)} \quad \text{since } n > 0$$

$$< \frac{17}{5+3n+3n^2}$$

$$< \frac{17}{3n+3n^2}$$

$$< \frac{17}{3n}$$

$$< \frac{17}{n}$$

$$< \varepsilon \quad \text{when } n > \frac{17}{\varepsilon}$$

side work: $\dfrac{17}{n} < \varepsilon$

$\Longleftrightarrow n > \dfrac{17}{\varepsilon}$

Therefore, $\dfrac{(1+2n)^2}{5+3n+3n^2} \longrightarrow \dfrac{4}{3}$ $\blacksquare$

6. $\{\sin(n)\}$ is bounded but does not converge.

7. $1, 1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, 5, \frac{1}{5}, \ldots$

convergent subsequence: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

divergent subsequence: $1, 2, 3, 4, 5, \ldots$

8. This is true. Let $\{a_n\}, \{b_n\} \subseteq \mathbb{R}$ be bounded.

Then $\exists M_1 \in \mathbb{R}$ st $|a_n| \leq M_1 \ \forall n \in \mathbb{N}$ and $\exists M_2 \in \mathbb{R}$ st $|b_n| \leq M_2 \ \forall n \in \mathbb{N}$.

Claim: $\{a_n + b_n\}$ is bounded by $M_1 + M_2$

$|a_n + b_n| \leq |a_n| + |b_n|$ Triangle Inequality

$\leq M_1 + M_2$ $\blacksquare$

$\{1-1/n^2\}$ are all bounded. But each of these three sequences is also convergent. Intuitively though, must this not be the case? If an increasing sequence is bounded, then the terms of the sequence would have to "pile up" somewhere; that is, the sequence would have to converge.

Theorem 2.3.3. *A bounded monotonic sequence converges.*

Proof. We prove the theorem for an increasing sequence; the decreasing sequence case is left for the exercises (Problem 5).

Assume that $\{x_n\}$ is bounded and increasing. To show that $\{x_n\}$ is convergent, we use the $\varepsilon - n_0$ definition of convergence, and to use this definition, we need to know the limit of the sequence. We determine the limit using the set $A = \{x_n \mid n \in \mathbb{N}\}$, the set of points in $\mathbb{R}$ consisting of the terms of the sequence $\{x_n\}$. Because $\{x_n\}$ is bounded, there is an $M > 0$ such that $|x_n| \leq M$ for all n, and this M is an upper bound for A. Hence A is a bounded nonempty set of real numbers and so has a least upper bound. Let $\alpha = \operatorname{lub} A$. This number α is the limit of $\{x_n\}$, which we now show.

Take any $\varepsilon > 0$. Since $\alpha = \text{lub} A$, there is an element $a \in A$ greater than $\alpha - \varepsilon$; that is, there is an integer n_0 such that $x_{n_0} = a > \alpha - \varepsilon$. But $\{x_n\}$ is increasing, so $x_{n_0} \leq x_n$ for all $n \geq n_0$, and α an upper bound of A implies that $x_n \leq \alpha$ for all n. Therefore, for $n \geq n_0$, $\alpha - \varepsilon < x_n \leq \alpha$, so $|x_n - \alpha| < \varepsilon$. □

Remark Note that, from the above proof, it follows immediately that, if α is the limit of an increasing sequence $\{x_n\}$, then $x_n \leq \alpha$ for all $n \in \mathbb{N}$. (Similarly, we have that the limit of a convergent decreasing sequence is a lower bound for the terms of the decreasing sequence.)

Example 2.3.4. Consider the sequence $\{x_n\}$, where x_n is the finite decimal expansion consisting of n ones after the decimal. Thus, $x_1 = 0.1$, $x_2 = 0.11$, $x_3 = 0.111$, and so on. This sequence is bounded by 0.2, for example, and is increasing. Hence the sequence converges. In fact, with this information, we can now easily compute the limit. Suppose $x_n \to a$. Then $x_{n+1} \to a$ also, and, using Theorem 2.2.5 and the equation $10x_{n+1} - x_n = 1$, we have

$$10a - a = \lim_{n\to\infty} (10x_{n+1} - x_n) = \lim_{n\to\infty} 1 = 1.$$

10. (proof sketch)

Let (X, d) be a metric space and $\{a_n\} \subseteq X$.

Assume $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$.

Say $a_{n_k} \to a$.

Let $\{b_n\}$ be a subsequence of $\{a_{n_k}\}$ and hence a subsequence of $\{a_n\}$. Then $b_n \to a$.

Let $\{b_{n_k}\}$ be a subsequence of $\{b_n\}$ and hence a subsequence of $\{a_n\}$. Then $b_{n_k} \to a$.

etc. ∎