Series

We've been talking a lot about sequences, which are infinite lists.

Example: $a_n = \frac{1}{n}$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

Today, we'll talk about series, which are infinite sums.

Example: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

If we add infinitely many positive numbers, won't the result be infinity?

Sometimes...

Example: $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

... but not always!

Example: $\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} + \cdots$

$= \frac{1}{3}$

Where is this number coming from?

Let $r \in \mathbb{R}$.

$\sum_{n=1}^{\infty} r^n$ is a **geometric series**

$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$ if $|r|<1$ $\quad$ ($r = \frac{1}{4}$ in the example)

What does this mean?

We can only compute the sum of finitely many terms at a time.

$S_2 = \sum_{n=1}^{2} \left(\frac{1}{4}\right)^n = \frac{1}{4} + \frac{1}{16} = 0.3125$

$S_3 = \sum_{n=1}^{3} \left(\frac{1}{4}\right)^n = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} = 0.328125$

$S_4 = \sum_{n=1}^{4} \left(\frac{1}{4}\right)^n = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} = 0.33203125$

$S_5 = \sum_{n=1}^{5} \left(\frac{1}{4}\right)^n = \underbrace{\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024}} = 0.3330078125$

These are called kth **partial sums** and are denoted S_k

The more terms we add (ie the larger k gets) the closer we get to the value of $\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$.

Notice that these partial sums form a sequence that appears to be looking more and more like $\frac{1}{3} = 0.33333\ldots$

$$\underset{S_2}{0.3125},\ \underset{S_3}{0.328125},\ \underset{S_4}{0.33203125},\ \underset{S_5}{0.3330078125}$$

Let (X, d) be a metric space.

Recall what it means for a sequence $\{a_n\} \subseteq X$ to converge to $a \in X$

$$a_n \to a \iff \forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ st } n \geq N \Rightarrow d(a_n, a) < \varepsilon$$

$\sum_{n=1}^{\infty} a_n$ converges to a (ie $\sum_{n=1}^{\infty} a_n = a$) if $\underbrace{S_k \longrightarrow a}$

To prove this, we have to figure out what S_k looks like

(Otherwise, $\sum_{n=1}^{\infty} a_n$ diverges.)

Goal: Prove that $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r} \iff |r|<1.$

This is called the Geometric Series Test.

Lemma 1 Let $r \in \mathbb{R} \setminus \{1\}$ and $K \in \mathbb{N}$.

$$S_K = \sum_{n=1}^{K} r^n = \frac{r(1-r^K)}{1-r}$$

can't divide by zero.

If $r=1$, then $\sum_{n=1}^{K} 1^n = \underbrace{1+1+1+\dots+1}_{K \text{ times}} = K$

Proof: (by induction)

$\boxed{\text{Step 1}}$ Show this is true for some particular $K \in \mathbb{N}$.

$(K=1)$ $\quad S_1 = \sum_{n=1}^{1} r^n = r = \frac{r(1-r)}{1-r}$

$(K=2)$ $\quad S_2 = \sum_{n=1}^{2} r^n = r + r^2 = \frac{r(1-r^2)}{1-r} = \frac{r(1+r)(1-r)}{1-r}$

$\boxed{\text{Step 2}}$ Let $N \in \mathbb{N}$. Assume $\sum_{n=1}^{N} r^n = \frac{r(1-r^N)}{1-r}$.

$\boxed{\text{Step 3}}$ Prove $\sum_{n=1}^{N+1} r^n = \frac{r(1-r^{N+1})}{1-r}$

$$\sum_{n=1}^{N+1} r^n = \underbrace{r + r^2 + r^3 + \dots + r^{N-1} + r^N}_{} + r^{N+1}$$

$$= \sum_{n=1}^{N} r^n \quad + r^{N+1}$$

$$= \frac{r(1-r^N)}{1-r} + r^{N+1} \quad \text{by the assumption in Step 2}$$

$$= \frac{r(1-r^N) + (1-r)r^{N+1}}{1-r}$$

$$= \frac{r - \cancel{r^{N+1}} + \cancel{r^{N+1}} - r^{N+2}}{1-r}$$

$$= \frac{r - r^{N+2}}{1-r}$$

$$= \frac{r(1-r^{N+1})}{1-r}$$

Therefore, $\sum_{n=1}^{K} r^n = \frac{r(1-r^K)}{1-r} \quad \forall K \in \mathbb{N}.$ $\blacksquare$

Our goal now is to prove $S_K = \frac{r(1-r^K)}{1-r}$ converges to $\frac{r}{1-r}$ if $|r| < 1$.

Properties of convergence

Let $a_n \to a$ and $b_n \to b$. Then

1. $a_n + b_n \to a + b$

2. $a_n - b_n \to a - b$

3. $a_n b_n \to ab$

4. $a_n / b_n \to a/b$

$b_n \neq 0$ $\forall n \in \mathbb{N}$

$b \neq 0$

So $\displaystyle\lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{r(1-r^k)}{1-r}$ only term involving k

$$= \frac{r\left(1 - \lim_{k\to\infty} r^k\right)}{1-r}$$

we need to find this

blue

Lemma 2 If $r \in \mathbb{R}$ st $|r|<1$, then $r^k \longrightarrow 0$.

Proof:

Case 1: $r=0$

Then we have a sequence of zeros, which converges to zero.

Case 2: $r \neq 0$ (so $0<|r|<1$)

Let $\varepsilon>0$.

$$\begin{aligned} d(r^k,0) &= |r^k-0| \\ &= |r^k| \\ &= |r|^k \\ &< \varepsilon \quad \text{if } k>\frac{\ln(\varepsilon)}{\ln|r|} \end{aligned}$$

side work: $|r|^k<\varepsilon$

$\Leftrightarrow \ln|r|^k<\ln(\varepsilon)$

$\Leftrightarrow k\ln|r|<\ln(\varepsilon)$

$\Leftrightarrow k>\dfrac{\ln(\varepsilon)}{\ln|r|}$

Therefore, $r^k \longrightarrow 0$ when $|r|<1$ ∎

green

Lemma 3 If $r=1$, then $r^k \to 1$.

Proof: $\{1^k\}$ is a sequence of ones, which converges to 1.

red

Lemma 4: If $r=-1$, then $\{r^k\}$ does not converge.

Proof: $\{(-1)^k\}$ looks like $-1, 1, -1, 1, -1, 1, \ldots$

Assume $\{(-1)^k\}$ converges to $a \in \mathbb{R}$.

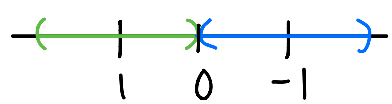
Let $\varepsilon = 1$.

Since $(-1)^k \to a$, $\exists N \in \mathbb{N}$ st $k \geq N \Rightarrow d((-1)^k, a) < \varepsilon = 1$

If k is odd, $(-1)^k = -1$, so we have that $d(-1, a) < 1$.

If k is even, $(-1)^k = 1$, so we have that $d(1, a) < 1$.

Thus, $a \in \mathbb{R}$ is within one unit of 1 and -1. $\rightarrow\leftarrow$



The green set and the blue set are disjoint.

Therefore, $\{(-1)^k\}$ does not converge. ∎

black squiggle

Lemma 5 If $r \in \mathbb{R}$ st $r > 1$, then $r^k \longrightarrow \infty$.

we didn't officially talk about what it means for a sequence to go to infinity, so we'll take this result for granted.

black solid

Lemma 6 If $r \in \mathbb{R}$ st $r < -1$, then r^k has no limit.

Example: $r = -2$

$\{r^k\} = \{-2, 4, -8, 16, -32, 64, \ldots\}$

We're ready now!

Theorem $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r} \iff |r| < 1.$

Proof:

Lemma 1 $\Rightarrow s_k = \frac{r(1+r^k)}{1-r}$

$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r} \iff s_k = \frac{r(1+r^k)}{1-r} \longrightarrow \frac{r}{1-r}$

$|r|<1$: Lemma 2 $\Rightarrow r^k \to 0$

$$\Rightarrow \frac{r(1+r^k)}{1-r} \to \frac{r(1+0)}{1-r} = \frac{r}{1-r}$$

$|r|=1$ (ie $r=\pm 1$): Lemma 3 $\Rightarrow \frac{r(1+r^k)}{1-r} \to \frac{1(1+1)}{1-1} = \frac{2}{0}$

undefined!

$$\sum_{n=1}^{\infty} 1^n = \sum_{n=1}^{\infty} 1 = 1+1+1+1+1+\dots = \infty$$

Lemma 4 $\Rightarrow \frac{r(1+r^k)}{1-r} \to \frac{(-1)(1+\dots)}{1-(-1)}$

$$\sum_{n=1}^{\infty} (-1)^n = -1+1-1+1-1+1-1+1-\dots$$

does not converge

$|r|>1$:

$r>1$: Lemma 5 $\Rightarrow r^k \to \infty$

$$\Rightarrow \frac{r(1+r^k)}{1-r} \to \frac{r(1+\infty)}{1-r} = \infty$$

$r<-1$: Lemma 6 $\Rightarrow$ the limit of r^k does not exist $\Rightarrow$ the limit of $\frac{r(1+r^k)}{1-r}$ does not exist

Therefore, $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r} \Leftrightarrow |r|<1$. $\blacksquare$