## Series (continued)

$\boxed{\text{Prove the } p\text{-series test.}}$

**Lemma 1** (nth term test, which we saw yesterday)
If $\{a_n\}$ does not converge to 0, then $\sum_{n=1}^{\infty} a_n$ diverges.

**Lemma 2** (integral test)
Let $a_n \geq 0 \ \forall n \in \mathbb{N}$. Let $f : [1, \infty) \to \mathbb{R}$ be such that $f(n) = a_n \ \forall n \in \mathbb{N}$.
If $f$ is positive and decreasing, then $\sum_{n=1}^{\infty} a_n$ converges $\iff$ $\int_1^{\infty} f(x)\, \mathrm{d}x$ equals a number.

**Theorem** (p-series test)
$\sum_{n=1}^{\infty} 1/n^p$ converges $\iff$ $p > 1$.

*Proof*:

**Step 1:** Consider the case when $p \leq 0$. Use Lemma 1 to show that $\sum_{n=1}^{\infty} 1/n^p$ diverges.

**Step 2:** Consider the case when $0 < p < 1$. Use the contrapositive of Lemma 2 to show that $\sum_{n=1}^{\infty} 1/n^p$ diverges.

<u>Step 2a:</u> State the contrapositive of Lemma 2.

<u>Step 2b:</u> Is $f(x) = 1/x^p$ positive and decreasing $\forall x \in [1, \infty)$ when $0 < p < 1$?

<u>Step 2c:</u> Compute $\int_1^{\infty} 1/x^p$ when $0 < p < 1$.

<u>Step 2d:</u> Conclude that $\sum_{n=1}^{\infty} 1/n^p$ diverges.

**Step 3:** Consider the case when $p = 1$.

**Step 4:** Consider the case when $p > 1$. Use Lemma 2 to show that $\sum_{n=1}^{\infty} 1/n^p$ converges.

<u>Step 4a:</u> Is $f(x) = 1/x^p$ positive and decreasing $\forall x \in [1, \infty)$ when $p > 1$?

<u>Step 4b:</u> Compute $\int_1^{\infty} 1/x^p$ when $p > 1$.

<u>Step 4c:</u> Conclude that $\sum_{n=1}^{\infty} 1/n^p$ converges.