

Continuity (cont.) Solutions

1. Let (X, d) be a metric space, $E \subseteq X$, and $p \in X$ a limit point of E (not necessarily in E). Prove there exists a sequence $\{p_n\}$ in E such that $p_n \neq p$ for all $n \in \mathbb{N}$ and $p_n \rightarrow p$.

Proof: For each $n \in \mathbb{N}$, consider the ball $B(p, \frac{1}{n})$. Since p is a limit point of E , there exists $q \neq p$ in E with $q \in B(p, \frac{1}{n})$. Denote this point p_n . Then $\{p_n\}$ is a sequence in E with $p_n \neq p$ for every $n \in \mathbb{N}$. We claim $p_n \rightarrow p$.

Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ with $\frac{1}{N} < \varepsilon$ (take $N > \frac{1}{\varepsilon}$). Then for any $n > N$,

$$d(p, p_n) < \frac{1}{n} \quad (\text{since } p_n \in B(p, \frac{1}{n}))$$
$$< \frac{1}{N} \quad \text{since } n > N$$
$$< \varepsilon.$$

Thus $p_n \rightarrow p$. □

2. Let (X, d_X) and (Y, d_Y) be metric spaces, $E \subseteq X$, $p \in X$ a limit point of E , $q \in Y$, and $f: E \rightarrow Y$. Prove $\lim_{x \rightarrow p} f(x) = q$ if and only if, for every sequence $\{p_n\}$ in E satisfying $p_n \neq p$ for all $n \in \mathbb{N}$ and $p_n \rightarrow p$, we then have $\lim_{n \rightarrow \infty} f(p_n) = q$.

Proof: (\Rightarrow) Suppose $\lim_{x \rightarrow p} f(x) = q$. Let $\{p_n\} \subseteq E$ with $p_n \neq p$ for all $n \in \mathbb{N}$ and $p_n \rightarrow p$. We wish to show $f(p_n) \rightarrow q$.

So let $\varepsilon > 0$. Since $f(x) \rightarrow q$ as $x \rightarrow p$, there exists $\delta > 0$ such that if $x \in E$ with $0 < d_X(x, p) < \delta$, then $d_Y(f(x), q) < \varepsilon$. And since $p_n \rightarrow p$, there exists $N \in \mathbb{N}$ such that $d_X(p_n, p) < \delta$ when $n > N$. In addition, $0 < d_X(p_n, p) < \delta$ since $p_n \neq p$. Thus, $n > N \Rightarrow 0 < d_X(p_n, p) < \delta \Rightarrow d_Y(f(p_n), q) < \varepsilon$. So $f(p_n) \rightarrow q$.

\Leftrightarrow Suppose $\lim_{x \rightarrow p} f(x) \neq q$. Then there exists

$\varepsilon > 0$ such that for any $\delta > 0$ you can find $x \in E$ with $0 < d_E(x, p) < \delta$ but $d_Y(f(x), q) > \varepsilon$.

Then for each $n \in \mathbb{N}$, denote such a point p_n for the associated $\delta = \frac{1}{n}$. Then by construction, $p_n \neq p$ for all $n \in \mathbb{N}$ and $p_n \rightarrow p$ but $f(p_n) \not\rightarrow q$ (the proof that $p_n \rightarrow p$ is similar to problem 1, and $f(p_n) \not\rightarrow q$ since for any $n \in \mathbb{N}$, $d_Y(f(p_n), q) > \varepsilon$). \blacksquare

3. Use problems 1 and 2 and the fact that limits of sequences are unique to conclude that the limit of a function is unique.

Proof: Suppose $\lim_{x \rightarrow p} f(x) = q$ and $\lim_{x \rightarrow p} f(x) = r$.

We wish to show $q = r$. By problem 1, there exists $\{p_n\} \subseteq E$ with $p_n \neq p$ for all $n \in \mathbb{N}$ and $p_n \rightarrow p$. Then by problem 2, $f(p_n) \rightarrow q$ and

$f(p_n) \rightarrow r$. But the limits of sequences are unique, so $q = r$ as desired. □