(worksheet)

1. $d(a,b) = 1$ if $a \neq b$ and $d(a,b) = 0$ if $a = b$

1. $d(a,b) > 0$ if $a \neq b$.

$a \neq b \Rightarrow d(a,b) = 1$ by definition of d

$\Rightarrow d(a,b) > 0$

If $a=b$, then $d(a,b)=0$.

This is true by definition of d.

2. $d(a,b) = d(b,a)$

If $a \neq b$, then $d(a,b) = 1 = d(b,a)$.

If $a=b$, then $d(a,b) = 0 = d(b,a)$.

3. $d(a,b) \leq d(a,c) + d(c,b)$

If $a \neq b$, then $d(a,b)=1$, so we need $d(a,c)+d(b,c)$ to equal 1 or 2, ie $\underbrace{d(a,c)+d(b,c) \neq 0}$.

Proof by contradiction: Assume this is not true, ie assume $d(a,c)+d(b,c) = 0$ and work to contradict a previous assumption.

$d(a,c)+d(b,c)=0$

$\Rightarrow d(a,c)=0$ and $d(b,c)=0$ since $d(a,c), d(b,c)\geq 0$

$\Rightarrow a=c$ and $b=c$

$\Rightarrow a=b \rightarrow\leftarrow$ we assumed $a\neq b$

Therefore, $d(a,c)+d(b,c)\neq 0$

$\Rightarrow d(a,c)+d(b,c)\geq 1$

$\Rightarrow d(a,c)+d(b,c)\geq d(a,b)$

If $a=b$, then $d(a,b)=0$.

$d(a,c), d(c,b)\geq 0$ by definition of d

$\Rightarrow d(a,c)+d(c,b)\geq 0$

$\Rightarrow d(a,c)+d(c,b)\geq d(a,b)$

Therefore, d is a metric on $\mathbb{R}$.
d is the discrete metric on $\mathbb{R}$. $\blacksquare$

2 $d(a,b) = |a^2 - b^2|$ fails condition 1

$a \neq b \Rightarrow d(a,b) > 0$

Counterexample: $a = 1 \quad b = -1$

$1 \neq -1$ but $d(1,-1) = |(1)^2 - (-1)^2|$

$= |1 - 1|$

$= 0$

Therefore, d is <u>not</u> a metric on $\mathbb{R}$.

③ $d(a,b) = |a^3 - b^3|$

1. $d(a,b) > 0$ if $a \neq b$

$a \neq b \Rightarrow a^3 \neq b^3$ since $f(x) = x^3$ is injective

$\Rightarrow a^3 - b^3 \neq 0$

$\Rightarrow |a^3 - b^3| \neq 0$

$\Rightarrow |a^3 - b^3| > 0$

$\Rightarrow d(a,b) > 0$

$d(a,b) = 0$ if $a = b$

$a = b \Rightarrow a^3 = b^3$

$\Rightarrow a^3 - b^3 = 0$

$\Rightarrow |a^3 - b^3| = 0$

$\Rightarrow d(a,b) = 0$

2. $d(a,b) = d(b,a)$

$$
\begin{aligned}
d(a,b) &= |a^3 - b^3| \\
&= |-(a^3 - b^3)| \\
&= |b^3 - a^3| \\
&= d(b,a)
\end{aligned}
$$

3. $d(a,b) \leq d(a,c) + d(c,b)$

$$
\begin{aligned}
d(a,b) &= |a^3 - b^3| \\
&= |a^3 + 0 - b^3| \\
&= |a^3 - c^3 + c^3 - b^3| \\
&\leq |a^3 - c^3| + |c^3 - b^3| \quad \text{triangle inequality in } \mathbb{R} \\
&= d(a,c) + d(c,b)
\end{aligned}
$$

Therefore, d is a metric on $\mathbb{R}$. $\blacksquare$

4 $d(a,b) = \max(a,b)$ fails condition 1

$d(a,b) > 0$ if $a \neq b$

Counterexample: $a = -2$, $b = -5$

$-2 \neq -5$ but $d(-2,-5) = \max(-2,-5)$

$= -2$

< 0

Therefore, d is <u>not</u> a metric on $\mathbb{R}$.

5 $d(a,b) = \max(|a|, |b|)$ fails condition 1

$d(a,b) = 0$ if $a = b$

Counterexample: $a = b = 7$

$d(7,7) = \max(|7|, |7|)$

$= \max(7,7)$

$= 7$

$\neq 0$

Therefore, d is not a metric on $\mathbb{R}$.

(6) $d(a, b) = \max(|x_1 - x_2|, |y_1 - y_2|)$

1. $d(a,b) > 0$ if $a \neq b$

$a \neq b \Rightarrow (x_1, y_1) \neq (x_2, y_2)$

$\Rightarrow x_1 \neq x_2$ or $y_1 \neq y_2$

$\Rightarrow x_1 - x_2 \neq 0$ or $y_1 - y_2 \neq 0$

$\Rightarrow |x_1 - x_2| \neq 0$ or $|y_1 - y_2| \neq 0$

$\Rightarrow |x_1 - x_2| > 0$ or $|y_1 - y_2| > 0$

$\Rightarrow \max(|x_1 - x_2|, |y_1 - y_2|) > 0$

$\Rightarrow d(a,b) > 0$

$a = b \Rightarrow (x_1, y_1) = (x_2, y_2)$

$\Rightarrow x_1 = x_2$ and $y_1 = y_2$

$\Rightarrow x_1 - x_2 = 0$ and $y_1 - y_2 = 0$

$\Rightarrow |x_1 - x_2| = 0$ and $|y_1 - y_2| = 0$

$$\Rightarrow \max(|x_1-x_2|, |y_1-y_2|)=0$$

$$\Rightarrow d(a,b)=0$$

2. $d(a,b)=d(b,a)$

$$d(a,b) = \max(|x_1-x_2|, |y_1-y_2|)$$

$$= \max(|-(x_1-x_2)|, |-(y_1-y_2)|)$$

$$= \max(|x_2-x_1|, |y_2-y_1|)$$

$$= d(b,a)$$

3. $d(a,b) \leq d(a,c)+d(c,b)$ $\quad c=(x_3,y_3)$

$$d(a,b) = \max(|x_1-x_2|, |y_1-y_2|)$$

$$= \max(|x_1+0-x_2|, |y_1+0-y_2|)$$

$$= \max(|x_1-x_3+x_3-x_2|, |y_1-y_3+y_3-y_2|)$$

$$\leq \max(|x_1-x_3|+|x_3-x_2|, |y_1-y_3|+|y_3-y_2|)$$

Triangle Inequality in $\mathbb{R}$

$$\leq \max(|x_1 - x_3|, |y_1 - y_3|) + \max(|x_3 - x_2|, |y_3 - y_2|)$$

$$\max(8+2, 1+8) = 10$$
$$\max(8,1) + \max(2,8)$$
$$= 8 + 8 = 16$$

$$= d(a,c) + d(c,b)$$

Therefore, d is a metric on $\mathbb{R}^2$
d is the maximum metric on $\mathbb{R}^2$ $\blacksquare$

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$d(a,b) = (x_1 - x_2)^2 + (y_1 - y_2)^2$

1. $d(a,b) > 0$ when $a \neq b$ ✓
   $d(a,b) = 0$ when $a = b$ ✓

2. $d(a,b) = d(b,a)$ ✓

fails condition 3 triangle inequality

Counterexample:

$a = (11, 10)$ $b = (1, 3)$ $c = (10, 2)$

$$d(a,b) = (11-1)^2 + (10-3)^2$$
$$= (10)^2 + (7)^2$$
$$= 100 + 49$$
$$= 149$$

$d(a,c)+d(c,b)$

$= (11-10)^2+(10-2)^2+(10-1)^2+(2-3)^2$

$= (1)^2+(8)^2+(9)^2+(-1)^2$

$= 1+64+81+1$

$= 65+82$

$= 147$

$d(a,b)=149 > 147 = d(a,c)+d(c,b)$

Therefore, d is <u>not</u> a metric on $\mathbb{R}^2$.

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$d(a,b) = \sqrt[3]{(x_1 - x_2)^3 + (y_1 - y_2)^3}$

fails condition 1

Counterexample: $a = (0,0)$
$b = (0,1)$

$$d(a,b) = \sqrt[3]{(0-0)^3 + (0-1)^3}$$
$$= \sqrt[3]{(-1)^3}$$
$$= -1 < 0$$

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9. (X, d) where $X = \mathbb{R}^2$ and $d(a,b) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ if a and b lie on the same line through the origin and $d(a,b) = \sqrt{(x_1)^2 + (y_1)^2} + \sqrt{(x_2)^2 + (y_2)^2}$ otherwise

d is a metric on $\mathbb{R}^2$.

10 $d(a,b) = 1$ fails condition 1

Counterexample: $a = (1,2)$

$$d(a,a) = 1 \neq 0$$

Therefore, d is <u>not</u> a metric on $\mathbb{R}^2$. ∎

One last thing:

Let $x, y \in \mathbb{R}$.

Triangle Inequality $|x+y| \leq |x| + |y|$

Reverse Triangle Inequality

$$\big|\, |x| - |y| \,\big| \leq |x-y|$$

note: this implies $|x| - |y| \leq |x-y|$

Proof of Reverse Triangle Inequality:

$$\begin{aligned} |x| &= |x+0| \\ &= |x-y+y| \\ &\leq |x-y| + |y| \quad \text{by Triangle Ineq.} \end{aligned}$$

$\Rightarrow |x| - |y| \leq |x-y|$

$$|y| = |y+0|$$
$$= |y-x+x|$$
$$\leq |y-x| + |x| \quad \text{by Triangle Ineq.}$$
$$= |x-y| + |x|$$
$$\Rightarrow |y| - |x| \leq |x-y|$$
$$\Rightarrow |x| - |y| \geq -|x-y|$$

Thus, $-|x-y| \leq |x| - |y| \leq |x-y|$

$$\Rightarrow \big|\, |x| - |y| \,\big| \leq |x-y| \qquad \blacksquare$$