**Cauchy Sequences**

1. Prove using the definition that the following sequence is Cauchy:

   $a_1 = 1$, $a_2 = 2$, and $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$ for $n \geq 3$.

   *Hint: Prove by induction that for* $n \geq 2$, $|a_n - a_{n-1}| = \frac{1}{2^{n-2}}$ *(if you do not know what induction is, just take this as a fact), and use the fact that* $1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^k} < 2$ *for every* $k \geq 0$.

Are the following metric spaces complete?

2. The interval $(0, 1)$ under the usual metric $d(a, b) = |b - a|$

3. The integers $\mathbb{Z}$ under the usual metric $d(m, n) = |m - n|$

Let $\{a_n\}$ be a Cauchy sequence in a metric space $(X, d)$. Prove the following.

4. $\{a_n\}$ is bounded.
5. If $\{a_n\}$ has a convergent subsequence, then $\{a_n\}$ converges.
6. Use Problems 4 and 5 and the Bolzano-Weirstrass Theorem to prove $\mathbb{R}$ is complete.