1. Prove by induction that $\sum_{n=1}^{k} n = 1+2+3+\dots+k = \frac{k(k+1)}{2}$.

Proof:

$\boxed{\text{Step 1}}$ Show this is true for some particular $k \in \mathbb{N}$.

$k=1$: $\sum_{n=1}^{1} n = 1 = \frac{1(1+1)}{2} = \frac{2}{2}$

$k=2$: $\sum_{n=1}^{2} n = 1+2 = 3 = \frac{2(2+1)}{2}$

$\boxed{\text{Step 2}}$ Let $N \in \mathbb{N}$. Assume $\sum_{n=1}^{N} n = \frac{N(N+1)}{2}$

$\boxed{\text{Step 3}}$ Prove $\sum_{n=1}^{N+1} n = \frac{(N+1)(N+1+1)}{2} = \frac{(N+1)(N+2)}{2}$

$$\sum_{n=1}^{N+1} n = \underbrace{1+2+3+\dots+(N-1)+N}+(N+1)$$

$$= \sum_{n=1}^{N} n + (N+1)$$

$$= \frac{N(N+1)}{2} + (N+1) \quad \text{by step 2 assumption}$$

$$= \frac{N(N+1)+2(N+1)}{2}$$

$$= \frac{(N+1)(N+2)}{2}$$

Therefore, $\sum_{n=1}^{k} n = \frac{k(k+1)}{2} \quad \forall k \in \mathbb{N}$. ∎

2. Lemma 4: If $r=-1$, then $\{r^k\}$ does not converge.

Proof: $\{(-1)^k\}$ looks like $-1, 1, -1, 1, -1, 1, \ldots$

Assume $\{(-1)^k\}$ converges to $a \in \mathbb{R}$.

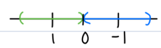
Let $\varepsilon = 1$.

Since $(-1)^k \to a$, $\exists N \in \mathbb{N}$ st $k \geq N \Rightarrow d((-1)^k, a) < \varepsilon = 1$

If k is odd, $(-1)^k = -1$, so we have that $d(-1, a) < 1$.

If k is even, $(-1)^k = 1$, so we have that $d(1, a) < 1$.

Thus, $a \in \mathbb{R}$ is within one unit of 1 and -1. $\rightarrow\leftarrow$



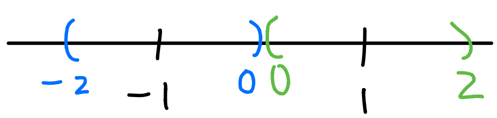
The green set and the blue set are disjoint.

Therefore, $\{(-1)^k\}$ does not converge. ∎

← What other values of ε could we have used to reach a contradiction?

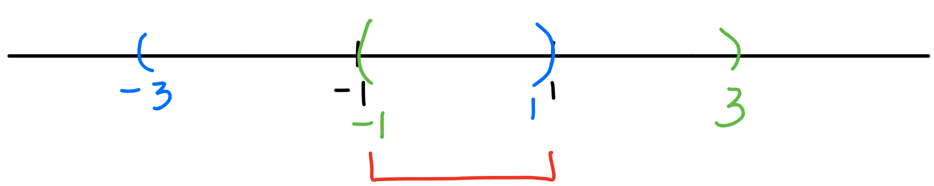
Answer: We need ε to be such that $d(-1, a) < \varepsilon$ and $d(1, a) < \varepsilon$ is not possible.

$\varepsilon = 1$ works because $(-2, 0)$ and $(0, 2)$ are disjoint.



If we let $\varepsilon = 2$, there is no contradiction.

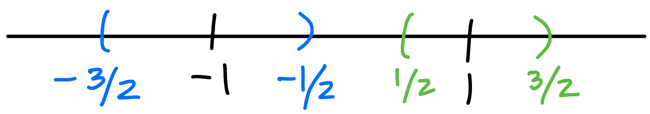
$d(-1, a) < 2$ and $d(1, a) < 2$



a could be any number in $(-1, 1)$ where $(-3, 1)$ and $(-1, 3)$ overlap

If we let $\varepsilon = 1/2$, there is a contradiction because

$(-3/2, -1/2)$ and $(1/2, 3/2)$ are disjoint.



In general, any ε st $0 < \varepsilon \leq 1$ will lead to a contradiction.

3. Prove that $\left\{ \frac{\sin(k\pi/2)}{8} \right\}$ does not converge.

Proof:

model

Lemma 4: If $r=-1$, then $\{r^k\}$ does not converge.

Proof: $\{(-1)^k\}$ looks like $-1, 1, -1, 1, -1, 1, \ldots$

Assume $\{(-1)^k\}$ converges to $a \in \mathbb{R}$.

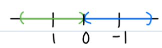
Let $\varepsilon = 1$.

Since $(-1)^k \to a$, $\exists N \in \mathbb{N}$ st $k \geq N \Rightarrow d((-1)^k, a) < \varepsilon = 1$

If k is odd, $(-1)^k = -1$, so we have that $d(-1, a) < 1$.

If k is even, $(-1)^k = 1$, so we have that $d(1, a) < 1$.

Thus, $a \in \mathbb{R}$ is within one unit of 1 and -1. $\rightarrow\leftarrow$



The green set and the blue set are disjoint.

Therefore, $\{(-1)^k\}$ does not converge. ∎

k	1	2	3	4	5	6	7	8	9	10	11	12	$\ldots$
$\frac{\sin(k\pi/2)}{8}$	$1/8$	0	$-1/8$	0	$1/8$	0	$-1/8$	0	$1/8$	0	$-1/8$	0	$\ldots$

Assume $\left\{ \frac{\sin(k\pi/2)}{8} \right\}$ converges to $a \in \mathbb{R}$.

Let $\varepsilon = 1/16$.

Since $\frac{\sin(k\pi/2)}{8} \to a$ $\exists N \in \mathbb{N}$ st

$$k \geq N \Rightarrow d\left(\frac{\sin(k\pi/2)}{8}, a\right) < \varepsilon = 1/16$$

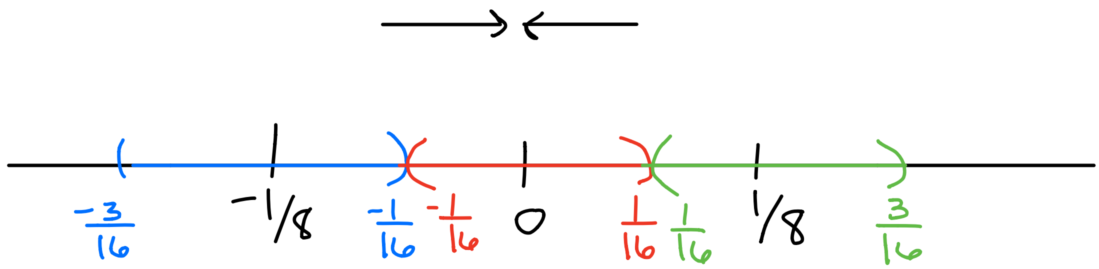
If k is odd, $\frac{\sin(k\pi/2)}{8} = \pm 1/8$, so we have that

$d(1/8, a) < 1/16$ and $d(-1/8, a) < 1/16$.

If k is even, $\frac{\sin(k\pi/2)}{8} = 0$, so we have that

$d(0,a) < 1/16$.

Thus, a is within $1/16$ of a unit of $1/8$, $-1/8$, and 0.



The blue set, the red set, and the green set are disjoint.

Therefore, $\left\{\frac{\sin(k\pi/2)}{8}\right\}$ does not converge. $\blacksquare$

4. Prove that $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$ converges.

a) What does s_K look like?

Answer:

$$s_1=\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}=1-\frac{1}{\sqrt{2}}=1-\frac{1}{\sqrt{1+1}}$$

$$s_2=1-\cancel{\frac{1}{\sqrt{2}}}+\cancel{\frac{1}{\sqrt{2}}}-\frac{1}{\sqrt{3}}=1-\frac{1}{\sqrt{3}}=1-\frac{1}{\sqrt{2+1}}$$

$$s_3=1-\cancel{\frac{1}{\sqrt{3}}}+\cancel{\frac{1}{\sqrt{3}}}-\frac{1}{\sqrt{4}}=1-\frac{1}{\sqrt{4}}=1-\frac{1}{\sqrt{3+1}}$$

$$s_4=1-\cancel{\frac{1}{\sqrt{4}}}+\cancel{\frac{1}{\sqrt{4}}}-\frac{1}{\sqrt{5}}=1-\frac{1}{\sqrt{5}}=1-\frac{1}{\sqrt{4+1}}$$

In general, $S_K=1-\frac{1}{\sqrt{K+1}}$

b) show that $\{s_K\}$ converges.

Proof: I claim $\lim_{K\to\infty} S_K=\lim_{K\to\infty}\left(1-\frac{1}{\sqrt{K+1}}\right)=1$

Let $\varepsilon>0$.

$$d\left(1-\frac{1}{\sqrt{K+1}},1\right)=\left|1-\frac{1}{\sqrt{K+1}}-1\right|$$
$$=\left|\frac{1}{\sqrt{K+1}}\right|$$

$$= \frac{1}{\sqrt{k+1}}$$

$$< \varepsilon \quad \text{if} \quad k > \frac{1}{\varepsilon^2} - 1$$

side work: $\frac{1}{\sqrt{k+1}} < \varepsilon$

$$\iff \frac{1}{\varepsilon} < \sqrt{k+1}$$

$$\iff \frac{1}{\varepsilon^2} < k+1$$

$$\iff \frac{1}{\varepsilon^2} - 1 < k$$

$$\iff k > \frac{1}{\varepsilon^2} - 1$$

Thus, $S_k \to 1$.

Therefore, $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$ converges. ∎