Properties of convergence

Let $a_n \to a$ and $b_n \to b$. Then

1. $a_n + b_n \to a+b$

2. $a_n - b_n \to a-b$

3. $a_n b_n \to ab$

4. $a_n / b_n \to a/b$

$b_n \neq 0$
$\forall n \in \mathbb{N}$

$b \neq 0$

Partial Proof: (#2)

$a_n \to a \Rightarrow \forall \varepsilon > 0 \ \exists N_1 \in \mathbb{N}$ st $n \geq N_1 \Rightarrow |a_n - a| < \varepsilon/2$

$b_n \to b \Rightarrow \forall \varepsilon > 0 \ \exists N_2 \in \mathbb{N}$ st $n \geq N_2 \Rightarrow |b_n - b| < \varepsilon/2$

$$\begin{aligned} |a_n - b_n - (a-b)| &= |a_n - b_n - a + b| \\ &= |a_n - a + b - b_n| \\ &\leq |a_n - a| + |b - b_n| \\ &= |a_n - a| + |b_n - b| \end{aligned}$$

$< \varepsilon/2 + \varepsilon/2$ when $n \geq N_1$ and $n \geq N_2$

ie when $n \geq \max(N_1, N_2)$

$= \varepsilon$

Therefore, $a_n - b_n \longrightarrow a - b$. ∎

Theorem | If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \longrightarrow 0$.

Proof: Assume $\sum_{n=1}^{\infty} a_n$ converges. Then the sequence $\{s_k\}$ of kth partial sums converges, say $s_k \longrightarrow c$.

$s_1 = a_1$

$s_2 = a_1 + a_2 = s_1 + a_2$

$s_3 = a_1 + a_2 + a_3 = s_2 + a_3$

In general,

$$s_k = s_{k-1} + a_k \Rightarrow \underset{\substack{\downarrow \\ c}}{s_k} - \underset{\substack{\downarrow \\ c}}{s_{k-1}} = a_k$$

$\Rightarrow a_k \longrightarrow 0$
by convergence property 2. ∎

Contrapositive: If a_n does not converge to zero, then $\sum_{n=1}^{\infty} a_n$ does not converge (ie it diverges)

This is commonly known as the **nth term test.**

Example: $\sum_{n=1}^{\infty} \frac{2n^2+1}{3n^2}$ $\quad a_n = \frac{2n^2+1}{3n^2} \longrightarrow \frac{2}{3} \neq 0$

So $\sum_{n=1}^{\infty} \frac{2n^2+1}{3n^2}$ diverges

Converse: If $a_n \to 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

False! Counterexample: $\sum_{n=1}^{\infty} \frac{1}{n}$

$a_n = \frac{1}{n} \longrightarrow 0$ but $\underbrace{\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}}_{\text{Let's prove this!}}$

Theorem 2 (direct comparison test) Let $0 \leq a_n \leq b_n \ \forall n \in \mathbb{N}$.

If $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

Idea: $\quad 0 \leq a_n \leq b_n \ \forall n \in \mathbb{N}$

$$\Rightarrow \sum_{n=1}^{\infty} 0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$$

$$\Rightarrow 0 \leq \sum_{n=1}^{\infty} a_n \leq B \quad \left(\text{assuming } \sum_{n=1}^{\infty} b_n \text{ converges to } B\right)$$

Proof: Let $S_K = \sum_{n=1}^{K} a_n$ and $T_K = \sum_{n=1}^{K} b_n$.

$\boxed{\text{Recall, a bounded monotonic sequence converges}}$

$a_n \geq 0 \ \forall n \in \mathbb{N} \Rightarrow \boxed{\{S_K\} \text{ is increasing.}}$

$b_n \geq 0 \ \forall n \in \mathbb{N} \Rightarrow \{T_K\}$ is increasing.

$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow T_K \to T$ (for some value T)

Since $\{T_K\}$ is increasing, $T_K \leq T \ \forall K \in \mathbb{N}$.

$a_n \leq b_n \ \forall n \in \mathbb{N} \Rightarrow S_K \leq T_K \ \forall K \in \mathbb{N}$

Thus, $S_K \leq T_K \leq T \ \forall K \in \mathbb{N}$

$\Rightarrow S_K \leq T \ \forall K \in \mathbb{N}$

$\Rightarrow \boxed{\{S_K\} \text{ is bounded}}$

Hence, $\{S_K\}$ converges.

Therefore, $\sum_{n=1}^{\infty} a_n$ converges. ∎

Theorem 3 (Contrapositive) Let $0 \leq a_n \leq b_n \ \forall n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$.

Claim: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof (of claim): Let $b_n = \frac{1}{n}$. To use Theorem 3, we need to find a_n st $0 \le a_n \le \frac{1}{n}$ $\forall n \in \mathbb{N}$.

$$
\begin{array}{lccccccccccccccccc}
\frac{1}{n}: & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} \\
 & = & = & > & = & > & > & > & = & > & > & > & > & > & > & > & = & > \\
a_n: & \boxed{1} & \boxed{\frac{1}{2}} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{64}
\end{array}
$$

$$
a_n: \boxed{1}\ \boxed{\tfrac{1}{2}}\ \underbrace{\boxed{\tfrac{1}{4}\ \tfrac{1}{4}}}_{\text{sum to } 1/2}\ \underbrace{\boxed{\tfrac{1}{8}\ \tfrac{1}{8}\ \tfrac{1}{8}\ \tfrac{1}{8}}}_{\text{sum to } 1/2}\ \underbrace{\boxed{\tfrac{1}{16}\ \tfrac{1}{16}\ \tfrac{1}{16}\ \tfrac{1}{16}\ \tfrac{1}{16}\ \tfrac{1}{16}\ \tfrac{1}{16}\ \tfrac{1}{16}}}_{\text{sum to } 1/2}\ \tfrac{1}{64}
$$

$0 \le a_n \le \frac{1}{n}$ $\forall n \in \mathbb{N}$

$\sum_{n=1}^{\infty} a_n = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots = \infty$ (diverges)

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. $\blacksquare$

Theorem $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff p > 1$.

Proof: worksheet

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a **p-series** and this theorem is called the p-series test.