Continuity (cont.)

1. Let (X, d) be a metric space, $E \subseteq X$, and $p \in X$ a limit point of E (not necessarily in E). Prove there exists a sequence $\{p_n\}$ in E such that $p_n \neq p$ for all $n \in \mathbb{N}$ and $p_n \to p$.

   Hint: For each $n \in \mathbb{N}$, consider the ball $B(p, \frac{1}{n})$. Since p is a limit point of E, what do you know about this ball? Use this to construct a sequence and show it converges to p.

2. Let (X, d_X) and (Y, d_Y) be metric spaces, $E \subseteq X$, $p \in X$ a limit point of E, $q \in Y$, and $f \colon E \to Y$. Prove $\lim\limits_{x \to p} f(x) = q$ if and only if, for every sequence $\{p_n\}$ in E satisfying $p_n \neq p$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} p_n = p$, we then have $\lim\limits_{n\to\infty} f(p_n) = q$.

   Hint: For the forward direction, assume $\lim\limits_{x \to p} f(x) = q$, and let $\varepsilon > 0$. Write out what it means that $\lim\limits_{x \to p} f(x) = q$ using that ε. Then also assume that you have such a sequence $\{p_n\}$ that converges to p. Write what that means using the δ you just found. Then put those together to show $\{f(p_n)\}$ converges to q.

   For the converse, prove the contrapositive: Assume $\lim\limits_{x \to p} f(x) \neq q$, and write out what that means. Use that to produce a sequence $\{p_n\}$ which converges to p, similar to problem 1, but $\{f(p_n)\}$ does not converge to q.

3. Use problems 1 and 2 and the fact that limits of sequences are unique to conclude that the limit of a function is unique.

   Hint: Assume $\lim\limits_{x \to p} f(x) = q$ and $\lim\limits_{x \to p} f(x) = r$. Problem 1 gives you the existence of a sequence $\{p_n\}$ which converges to p but is never equal to p. What does problem 2 tell you about $\{f(p_n)\}$? And how does the uniqueness of limits of sequences then tell you that $q = r$?