Metric Spaces

Definition Let X be a set. State the definition of a metric on d on X.

Fill in the blanks Let $a=(x_1,y_1)$ and $b=(x_2,y_2)$ be points in the plane $\mathbb{R}^2$. $d(a,b)=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ is the ____________ metric on $\mathbb{R}^2$ and $d(a,b)=|x_1-x_2|+|y_1-y_2|$ is the ____________ metric on $\mathbb{R}^2$.

True or false If $x,y\in\mathbb{R}$, then $|x+y|<|x|+|y|$.

Example Give an example of a metric space.

Proof Let $a,b\in\mathbb{R}$. Prove that $d(a,b)=|a-b|$ is a metric on $\mathbb{R}$.

Sequences

Definition Let (X,d) be a metric space, $p \in X$, and $r \in \mathbb{R}$ st $r > 0$. State the definition of the ball centered at p with radius r.

Fill in the blanks Let (X,d) be a metric space, $\{a_n\}$ be a sequence in X, and $a \in X$ be a point in X. We say $\{a_n\}$ converges to $a \iff \forall\, \varepsilon > 0$ $\exists N \in \mathbb{N}$ st n ___ $N \Rightarrow$ ______ $< \varepsilon$.

True or false Every convergent sequence is bounded.

Example Give an example of a monotonic sequence in $\mathbb{R}$.

Proof Prove that $1 - \frac{(-1)^n}{n} \longrightarrow 1$.

$\boxed{\text{Series}}$

Definition Let (X,d) be a metric space, $\{a_n\}$ be a sequence in X, and $a \in X$ be a point in X. State the definition of $\sum_{n=1}^{\infty} a_n = a$.

Fill in the blanks Let $r \in \mathbb{R}$. $\sum_{n=1}^{\infty} r^n$ is a

____________ series and converges to __________ if

$|r|<1$.

True or false $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

Example Give an example of a series that diverges.

Proof Prove by induction that $\sum_{n=1}^{K} n = 1+2+3+\dots+K = \frac{K(K+1)}{2}$

Ordered Sets and Bounds

Definition: Let $(X,<)$ be an ordered set, and $E \subseteq X$ a subset of X. State the definition of what it means for E to be bounded above.

Fill in the blanks: If X is a set, a relation $<$ on S is called an order if

(i) for $x, y \in X$ with $x \neq y$, then ________ of the following is true:

$$x < y \quad \text{or} \quad y < x$$

(ii) $<$ is ____________.

True or False: $\mathbb{R}$ has the greatest lower bound property.

Example: Give an example of a set which is bounded below

Proof: Let $E \subseteq \mathbb{R}$ be nonempty and bounded below. Let $-E = \{-x \mid x \in E\}$. Prove $-E$ is bounded above and that $\inf A = -\sup(-A)$.

Cauchy Sequences

Definition: Let $\{a_n\}$ be a sequence in a metric space (X,d). State the definition of what it means for $\{a_n\}$ to be Cauchy.

Fill in the blanks: A metric space where every __________ converges to a point in that space is called ________.

True / False: $\mathbb{Z}$ is complete.

Example: Give an example of a metric space and a Cauchy sequence which does not converge in that space.

Proof: Prove every Cauchy sequence in a metric space is bounded.

Continuity

Definition: If X is a metric space and $E \subseteq X$, what is the definition of a limit point of E?

Fill in the blanks: If $f: E \to Y$ where $E \subseteq X$ and X and Y are metric spaces, we say $\lim_{x \to p} f(x) = q$ if p is a ______ of E and for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in E$ with $d_X(x,p)$ ____, then $d_Y(f(x), q)$ ____.

True or False: The function $f(x) = \begin{cases} \frac{x^2+3x+2}{x+1}, & x \neq -1 \\ 2 & x = 1 \end{cases}$ is continuous on all of $\mathbb{R}$.

Example: Give an example of a function which is continuous on all of $\mathbb{R}$.

Proof: Prove that $\lim_{x \to 3} -2x+1 = -5$.