Casual Dip into Lie Theory

What is a Lie Algebra?

For a field k, $\mathfrak{g}$ is a Lie algebra if it's a k-algebra with bilinear map $[\cdot,\cdot]$ s.t.:

1) $[x,x]=0$

2) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$

3) $[x,y]=-[y,x]$

Examples: • $\mathfrak{gl}_n(k)$ $\quad \mathrm{Mat}_n(k)$; $[A,B]=AB-BA \in \mathfrak{gl}_n(k)$

• $\mathfrak{sl}_n(\mathbb{C}) = \{A \in \mathrm{Mat}_n(k) \mid \mathrm{Tr}(A)\}$; $[A,B]=AB-BA \in \mathfrak{sl}_n(k)$

A_{n-1}

$$\mathrm{Tr}(AB)=\mathrm{Tr}(A)\,\mathrm{Tr}(B)$$

$$\mathrm{Tr}(AB)-\mathrm{Tr}(BA)=0$$

• $\mathfrak{so}_8(\mathbb{C}) = \{A \in \mathrm{Mat}_8(k) \mid A^tM+MA=0\}$ $\quad M = \begin{bmatrix} 0 & I_4 \\ I_4 & 0 \end{bmatrix}$;

D_4

$D \sim \mathfrak{so}_{2n}$

$$\begin{aligned}[A,B]^tM - M[A,B] &= (AB-BA)^tM + M(AB-BA) \\ &= B^tA^tM - A^tB^tM + MAB - MBA \\ &= -B^tMA + A^tMB - A^tMB + B^tMA \\ &= 0 \end{aligned}$$

$\Rightarrow [A,B] \in \mathfrak{so}_8(\mathbb{C})$

Ok, but how do you work with them?

First you need to choose a Cartan Subalgebra, $\mathfrak{h}$.

A **CSA** is a maximal subalgebra of entirely semisimple elements.

An element is **semisimple** if its adjoint action is diagonalizable.

Examples: • diagonal matrices in $\mathfrak{sl}_n(\mathbb{C})$

• diagonal matrices in $\mathfrak{so}_8(\mathbb{C})$

Since diagonal, ad(A) is diagonal

Once we have one chosen, then we can talk about rep's.

- A **representation of $\mathfrak{g}$** (a.k.a $\mathfrak{g}$-module) is a $\Bbbk$ vector space V coupled with a Lie algebra homomorphism $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$.

- If $\mathfrak{g}$ semisimple, V f.d., the V splits into its **weight spaces**;

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu \quad ; \quad V_\mu = \{v \in V \mid hv = \mu(h)v \ \forall h \in \mathfrak{h}\}$$

- We define the **weights** of a representation are

$$\mathrm{wt}\, V = \{\mu \in \mathfrak{h}^* \mid V_\mu \neq 0\}$$

Super important example: The adjoint representation.

$$V = \mathfrak{g} \qquad \mathrm{ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$
$$x \mapsto [x, \cdot] = \mathrm{ad}_x(\cdot)$$

it makes $\mathfrak{g}$ a left module of itself.

What makes this rep. so important?

The non-zero weights are the **roots** of $\mathfrak{g}$.

Finding roots of $\mathfrak{sl}_n(\mathbb{C})$:

$$V_\mu = \{ A \in \mathfrak{sl}_n(\mathbb{C}) \mid [H, A] = \mu(H) A \ \forall H \in \mathfrak{h}\}.$$

Here $\mathfrak{h} = \{\text{Diagonal matrices with trace } 0\}$, so $\forall H \in \mathfrak{h}$, $\sum_{i=1}^{n} H_{ii} = 0$.

, so $[H, A] = HA - AH = (H_{ii} - H_{jj}) A_{ij}$ and $\mu(H) = (H_{ii} - H_{jj}) = (\varepsilon_i - \varepsilon_j) H$

where ε_i = n-tuple with 1 in the i^{th} coordinate, 0's elsewhere.

$\to \mathfrak{sl}_n$ roots are $\{\pm(\varepsilon_i - \varepsilon_j) \ \forall\ 1 \le i < j \le n\}$

Finding roots of $\mathfrak{so}_8(\mathbb{C})$:

$$V_\mu = \{ A \in \mathfrak{so}_8(\mathbb{C}) \mid [H,A] = \mu(H)A \ \forall H \in \mathfrak{h} \}.$$

Here $\mathfrak{h} = \{\text{Diagonal matrices s.t } H^t M + MH = 0\}$

$$H = \underset{\sim}{\begin{bmatrix} H^1 & 0 \\ 0 & H^2 \end{bmatrix}} \curvearrowright \quad H^t M = \begin{bmatrix} H^1 & 0 \\ 0 & H^2 \end{bmatrix}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} 0 & H^1 \\ H^2 & 0 \end{bmatrix}$$

$$MH = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} H^1 & 0 \\ 0 & H^2 \end{bmatrix} = \begin{bmatrix} 0 & H^2 \\ H^1 & 0 \end{bmatrix}$$

, so for $H^t M = -MH$,

$H^1 = -H^2$

or $H_{ii} = -H_{(i+4)(i+4)}$ for $1 \le i \le 4$.

$$[H,A] = HA - AH = \begin{cases} (H_{ii} - H_{jj})\, A_{ij} \\ (H_{ii} + H_{jj})\, A_{i(j+4)} \\ (-H_{ii} - H_{jj})\, A_{(i+4)j} \\ (-H_{ii} + H_{jj})\, A_{(i+4)(j+4)} \end{cases}$$

, so
$$\mu(H) = \begin{cases} (H_{ii} - H_{jj}) = (\varepsilon_i - \varepsilon_j)H \\ (H_{ii} + H_{jj}) = (\varepsilon_i + \varepsilon_j)H \\ (-H_{ii} - H_{jj}) = (-\varepsilon_i - \varepsilon_j)H \\ (-H_{ii} + H_{jj}) = (-\varepsilon_i + \varepsilon_j)H \end{cases}$$

, where $\varepsilon_i = \underbrace{4\text{-tuple}}$ with 1 in the i^{th} place

$\rightarrow \mathfrak{so}_8(\mathbb{C})$ roots are $\{(\pm\varepsilon_i \pm \varepsilon_j)\ \forall\ 1 \le i < j \le 4\}$

So, what do we do with roots?

Take the set of $\mathfrak{g}$'s roots to be R.

We can choose a basis of R so that all roots are in the $\mathbb{Z}^+$-span or $\mathbb{Z}^-$-span of this basis. These are simple roots, Q.

for $\mathfrak{sl}_n(\mathbb{C})$:

$$R = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \le i < j \le n\}$$

$$Q = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \le i \le n-1\}$$

$$\varepsilon_1 - \varepsilon_4 = \underbrace{(\varepsilon_1 - \varepsilon_2)}_{\alpha_1} + \underbrace{(\varepsilon_2 - \varepsilon_3)}_{\alpha_2} + \underbrace{(\varepsilon_3 - \varepsilon_4)}_{\alpha_3}$$

for $\mathfrak{so}_8(\mathbb{C})$:

$$R = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le 4\}$$

$$Q = \left\{\begin{matrix} \alpha_i = \varepsilon_i - \varepsilon_{i+1} & 1 \le i \le 3 \\ \alpha_4 = \varepsilon_3 + \varepsilon_4 & \end{matrix}\right\}$$

$$\varepsilon_2 + \varepsilon_3 = \underbrace{(\varepsilon_2 - \varepsilon_3)}_{\alpha_2} + \underbrace{(\varepsilon_3 - \varepsilon_4)}_{\alpha_3} + \underbrace{(\varepsilon_3 + \varepsilon_4)}_{\alpha_4}$$

Once we have simple roots, we'll take Q^+ to be the $\mathbb{Z}^+$-span of Q, and our positive roots will be $Q^+ \cap R = R^+$.

We can create a basis for $\mathfrak{g}$, the Chevalley basis, indexed by R^+.

$$\{x_\alpha^-, h_\alpha, x_\alpha^+ \mid \alpha \in R^+\}$$

negative root space $\quad$ positive root space

where $[x_\alpha^+, x_\alpha^-] = h_\alpha$

$$[h_\alpha, x_\beta^\pm] = \frac{2(\pm\beta,\alpha)}{(\alpha,\alpha)} x_\alpha^\pm \qquad \rightarrow (\beta,\alpha) = \mathrm{Tr}(\mathrm{ad}_{x_\beta}\mathrm{ad}_{x_\alpha})$$

length β projected onto α.

$$[x_\alpha^\pm, x_\beta^\pm] = c\, x_{\alpha+\beta}^\pm \text{ iff } \alpha+\beta \in R^+$$

$c \in \mathbb{C}$

From here, we'll define: $\omega_4(h_{\alpha_4}) = 1$

- fundamental weights: $\{\omega_1, \dots, \omega_n \mid \omega_i(h_{\alpha_j}) = \delta_{ij}\} \subset \mathfrak{h}^*$
- integral weights: $\{\mathbb{Z}\text{-span of fundamental weights}\}$
- dominant integral wt.s: $\{\mathbb{Z}^+\text{-span of fundamental weights}\}$

Since our roots are non-zero weights of the adjoint representation, we can express our roots with these fundamental weights.

$\mathfrak{sl}_n(\mathbb{C})$:

$\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 < i < n-1\}$

$h_{\alpha_i} = E_{ii} - E_{(i+1)(i+1)}$ (E_{ij} is $n \times n$ matrix with 1 in ij^{th} position 0's else)

$\alpha_i(h_i) = 1 - (-1) = 2$

$\alpha_i(h_{i+1}) = 0 - 1 = -1$

$\alpha_i(h_{i-1}) = -1 - 0 = -1$

$\alpha_i(h_j) = 0 \quad j \notin [i-1, i+1]$

$\Rightarrow \left[\alpha_i = -\omega_{i-1} + 2\omega_i - \omega_{i+1}\right]$

$\mathfrak{so}_8(\mathbb{C})$:

$\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 < i < 3\} \cup \{\alpha_4 = \varepsilon_3 + \varepsilon_4\}$

$h_{\alpha_i} = (E_{ii} - E_{(i+4)(i+4)}) - (E_{(i+1)(i+1)} - E_{(i+5)(i+5)}) \quad 1 < i < 3$

$h_{\alpha_4} = (E_{33} - E_{77}) + (E_{44} - E_{88})$

$\alpha_i(h_i) = \begin{cases} 1 - (-1) = 2 & 1 < i < 3 \\ 1 + 1 = 2 & i = 4 \end{cases}$

$\alpha_i(h_{i+1}) = \begin{cases} 0 - 1 = -1 & 1 < i < 2 \\ 1 - 1 = 0 & i = 3 \end{cases}$

$\alpha_i(h_{i-1}) = \begin{cases} -1 - 0 = -1 & 2 < i < 3 \\ 1 - 1 = 0 & i = 4 \end{cases}$

$\alpha_2(h_{\alpha_4}) = 0 - 1 = -1$

$\alpha_4(h_{\alpha_2}) = -1 + 0 = -1$

$$\left[\begin{array}{ll} \alpha_1 = 2\omega_1 - \omega_2 & \alpha_3 = -\omega_2 + 2\omega_3 \\ \alpha_2 = -\omega_1 + 2\omega_2 - \omega_3 - \omega_4 & \alpha_4 = -\omega_2 + 2\omega_4 \end{array}\right]$$

So what does this have to do with reps. in general?

Remember that a rep. is a vector space V with $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$? And if $\mathfrak{g}$ semisimple, V f.d., $V = \bigoplus_{\mu} V_{\mu}$?

So ~~($\forall v \in V$, $\exists$ a weight μ s.t.)~~ Take a $v \in V_{\mu}$. $\Rightarrow$ $hv = \mu(h)v \;\forall h \in \mathfrak{h}$.

If I take $hv = (\text{weight of } v)v$.

then
$$h(x_{\alpha}^{\pm} v) = [h, x_{\alpha}^{\pm}]v + x_{\alpha}^{\pm}(hv)$$
$$= \pm\alpha(h)\, x_{\alpha}^{\pm} v + x_{\alpha}^{\pm} \mu(h) v$$
$$= \mu \pm \alpha(h)(x_{\alpha}^{\pm} v)$$

What this boils down to is that action of x_{α}^{+} (x_{α}^{-}) on a vector in my rep. moves my vector "up" α weight spaces ("down" α weight spaces).

So knowing our Chevalley basis and weight expression of our roots tells us exactly the way $\mathfrak{g}$ will interact with it's rep's.

Alright, and?

Well, there is a specific kind of representation that I concern myself with: Highest Weight Modules.

V is a highest weight module if $\exists\, v \in V$ s.t

- $V = \mathcal{U}(\mathfrak{g})\, v$ ($\mathcal{U}(\mathfrak{g})$ is the algebra of words of $\mathfrak{g}$ reducible with $\mathfrak{g}$'s bracket)
- $x_\alpha^+ v = 0 \;\; \forall\, \alpha \in R^+$
- λ is the highest weight, $v = V_\lambda$

Example: If $\mathfrak{g}$ is simple, its adjoint rep. is a highest weight module, with highest weight = highest root.

$\alpha \in R^+$ s.t $\forall\, \beta \in R^+$, $\alpha + \beta \notin R^+$.

But the algebras I look at these modules over aren't just simple Lie algebras.

So what algebras do you care about?

I work with Current algebras of simple Lie algebras:

For a simple Lie algebra $\mathfrak{g}$, that's

$$\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$$

This is a Lie algebra in its own right:

$$[x \otimes t^m, y \otimes t^n] = [x,y] \otimes t^{m+n} \quad \forall x,y \in \mathfrak{g}.$$

In 2001, Chari and Pressley introduced local Weyl Modules, which are highest weight modules for these algebras:

$W_{loc}(\lambda) = U(\mathfrak{g}[t])w_\lambda$ with relations:

$$\left.\begin{array}{l} \cdot\ (x_\alpha^+ \otimes 1)w_\lambda = 0 \\ \cdot\ (h_\alpha \otimes t^r)w_\lambda = \delta_{r,0}\,\lambda(h_\alpha)\,w_\lambda \\ \cdot\ \underbrace{(x_\alpha^- \otimes 1)^{\lambda(h_\alpha)+1} w_\lambda = 0}_{\text{can only drop so low}} \end{array}\right\} \forall \alpha \in R^+$$

These modules give rise to another type of highest-weight module; the Demazure Modules. We'll use a Theorem of Chari and Venkatesh as the definition:

For $\ell \in \mathbb{N}$, $\lambda \in P^+$, $D(\ell,\lambda)$ is the quotient of $W_{loc}(\lambda)$ with additional relations:

$$(x_\alpha^- \otimes t^{s_\alpha}) w_\lambda = 0$$

$$(x_\alpha^- \otimes t^{s_\alpha - 1})^{m_\alpha + 1} w_\lambda = 0 \quad \text{if } m_\alpha < \frac{2\ell}{(\alpha,\alpha)}$$

$$\underbrace{\lambda(h_\alpha) = \frac{2\ell}{(\alpha,\alpha)}(s_\alpha - 1) + m_\alpha; \; 0 < m_\alpha \leq \frac{2\ell}{(\alpha,\alpha)}}$$

this is a modified division algorithm, so think about these modules as trying to split λ into ℓ pieces.

What do you do with these modules?

Essentially, for all simply-laced algebras, the dimensions of the weight spaces of $W_{loc}(\lambda) \cong D(1,\lambda)$ are known, but not so much for $\ell > 1$. Biswal-Chari-Shereen-Wand found a way to describe the dimensions of $D(2,\lambda)$'s weight spaces in terms of $D(1,\lambda)$'s for $\mathfrak{sl}_n(\mathbb{C})[t]$ last year, and so I am trying to do the same for $\mathfrak{so}_{2n}(\mathbb{C})[t]$.